On homogenization of space-time dependent and degenerate random flows

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Abstract

We study a diffusion with time-dependent random coefficients. The diffusion coefficient is allowed to degenerate. We prove an invariance principle when this diffusion is supposed to be controlled by another one with time independent coefficients.

1 Introduction

We want to establish an invariance principle for a diffusive particle in a random flow described by the following Stochastic Differential Equation (SDE)

$$X_t^\omega = x + \int_0^t b\left(r, X_r^\omega, \omega\right)\, dr + \int_0^t \sigma\left(r, X_r^\omega, \omega\right)\, dB_r,$$

where B is a d-dimensional Brownian motion and σ, b are stationary random fields. b is defined in such a way that the generator at time t of the diffusion coincides on smooth functions with

$$(1) \qquad L^\omega = (1/2)e^{2V(x,\omega)}\mathrm{div}_x\big(e^{-2V(x,\omega)}[a(t,x,\omega) + H(t,x,\omega)]\nabla_x\ \big)\,.$$

Here $a(t,x,\omega)$ is equal to $\sigma\sigma^*(t,x,\omega)$. V and H are stationary random fields, V is bounded and H antisymmetric.
We will then be in position to study the effective diffusion on a macroscopic scale of the following convection-diffusion equation

$$(2) \qquad \partial_t z(t,x,\omega) = (1/2)\mathrm{Trace}[a\triangle_{xx}z](t,x,\omega) + b\cdot\nabla_x z(t,x,\omega),$$

with certain initial condition. We will prove that, in probability with respect to ω,

$$\lim_{\varepsilon\to 0} z(t/\varepsilon^2, x/\varepsilon, \omega) = \overline{z}(t,x)$$

where $\overline{z}$ is the solution of a deterministic equation

$$(3) \qquad \partial_t\overline{z}(t,x) = \mathrm{Trace}[A\triangle_{xx}\overline{z}](t,x).$$

A is a constant matrix - the matrix of so-called effective coefficients.

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Homogenization problems have been extensively studied in the case of periodic flows (cf. [1], [16], [17], and many others). The study of random flows (see [14], [15], [20], and many others) spread rapidly thanks to the techniques of the *environment as seen from the particle* introduced by Kipnis and Varadhan in [7], at least in the case of time independent random flows. Recently, there have been results going beyond these techniques in the case of isotropic coefficients which are small perturbations of Brownian motion (see [21]). But there are only a few works in the case of space-time dependent random flows (see [10] or [11] for instance in the case $\sigma = \mathrm{Id}$). A quenched version of the invariance principle is stated in [2] provided that the diffusion coefficient satisfies a strong uniform non-degeneracy assumption. In this case, the regularizing properties of the heat kernel are widely used to face with the non-reversibility of the underlying processes. Some results stated in Markovian flows are also established in [3] or [4].
The novelty of this work lies in the ergodic and regularizing properties required on the coefficients, which are not far from being minimal. The only restriction is the control of the diffusion process with an ergodic and time independent one. As a consequence, this work includes the static case where all the coefficients do not depend on time. Moreover, these assumptions allow the diffusion matrix to degenerate. Typically it can degenerate in certain directions or vanish on subsets of null measure but cannot totally reduce to zero on an open subset of $\mathbb{R}^d$. However, considering such strong degeneracies remains a quite open problem for random stationary coefficients (for recent advances in the static periodic case, see [16]).

We will outline now the main ideas of the proof. Our goal is to show that the rescaled process

$$\varepsilon X^{\omega}_{t/\varepsilon^2} = \varepsilon \int_0^{t/\varepsilon^2} b\left(r, X^{\omega}_r, \omega\right) \, ds + \varepsilon \int_0^{t/\varepsilon^2} \sigma\left(r, X^{\omega}_r, \omega\right) \, dB_s$$

converges in law to a Brownian motion with a certain positive covariance matrix. The general strategy (see [8]) consists in finding an approximation of the first term on the right-hand side by a family of martingales and then in applying the central limit theorem for martingales. To find such an approximation, we look at the environment as seen from the particle

$$Y_t = \tau_{t, X^{\omega}_t} \omega,$$

where $\{\tau_{t,x}\}$ is a group of measure preserving transformation on a random medium Ω (see Definition 2.1). Thanks to the particular choice of the drift, an explicit invariant measure can be found for this Markov process. The ergodicity is ensured by the geometry of the diffusion coefficient σ (see Assumptions 2.3 and 2.4). The approximation that we want to find leads to study the equation ($\lambda > 0$)

$$\lambda \boldsymbol{u}_\lambda - (\boldsymbol{L} + D_t)\boldsymbol{u}_\lambda = \boldsymbol{b} \tag{4}$$

where $\boldsymbol{L} + D_t$ coincides with the generator of the process Y on a certain class of functions (the term D_t is due to the time evolution and $\boldsymbol{L}$ is an unbounded operator on the medium Ω associated to (1)). Here are arising the difficulties resulting from the time dependence. Due to the term D_t, the Dirichlet form associated to $\boldsymbol{L} + D_t$ does not satisfy any sector condition (even weak). However, for a suitable function $\boldsymbol{b}$, (4) can be solved with the help of an approximating sequence of Dirichlet forms with weak sector condition. Then, usual techniques used in the static case fall short of establishing the so-called sublinear growth of the correctors $\boldsymbol{u}_\lambda$. To get round this difficulty, regularizing properties of the heat kernel are used in [2], [10] or [11]. Here the degeneracies of the diffusion coefficient prevents us from using such arguments. The strategy here consists in separating the time and spatial evolutions (see Assumption 2.3). We introduce a new operator $\widetilde{\boldsymbol{S}}$ whose coefficients do not depend on time. Then the spectral calculus linked to the normal operator $\widetilde{\boldsymbol{S}} + D_t$ will be determining to establish the desired estimates for the solution $\boldsymbol{v}_\lambda$ of the equation

$$\lambda \boldsymbol{v}_\lambda - (\widetilde{\boldsymbol{S}} + D_t)\boldsymbol{v}_\lambda = \boldsymbol{b}.$$

Finally, with perturbation methods, we show that these estimates remain valid for the correctors $\boldsymbol{u}_\lambda$.

2 Notations, Setup and Main Result

Let us first introduce a random medium

Definition 2.1. *Let $(\Omega, \mathcal{G}, \mu)$ be a probability space and $\{\tau_{t,x}; (t,x) \in \mathbb{R} \times \mathbb{R}^d\}$ a stochastically continuous group of measure preserving transformations acting ergodically on Ω:*

1) $\forall A \in \mathcal{G}, \forall (t,x) \in \mathbb{R} \times \mathbb{R}^d$, $\mu(\tau_{t,x}A) = \mu(A)$,

2) If for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $\tau_{t,x}A = A$ then $\mu(A) = 0$ or 1,

3) For any measurable function $\boldsymbol{g}$ on $(\Omega, \mathcal{G}, \mu)$, the function $(t,x,\omega) \mapsto \boldsymbol{g}(\tau_{t,x}\omega)$ is measurable on $(\mathbb{R} \times \mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R} \times \mathbb{R}^d) \otimes \mathcal{G})$.

In what follows we will use the bold type to denote a function $\boldsymbol{g}$ from Ω into $\mathbb{R}$ (or more generally into $\mathbb{R}^n$, $n \geq 1$) and the unbold type $g(t,x,\omega)$ to denote the associated representation mapping $(t,x,\omega) \mapsto \boldsymbol{g}(\tau_{t,x}\omega)$. The space of square integrable functions on $(\Omega, \mathcal{G}, \mu)$ is denoted by $L^2(\Omega)$, the usual norm by $|\cdot|_2$ and the corresponding inner product by $(\cdot\,,\cdot)_2$. Then, the operators on $L^2(\Omega)$ defined by $T_{t,x}\boldsymbol{g}(\omega) = \boldsymbol{g}(\tau_{t,x}\omega)$ form a strongly continuous group of unitary maps in $L^2(\Omega)$. Each function $\boldsymbol{g}$ in $L^2(\Omega)$ defines in this way a stationary ergodic random field on $\mathbb{R}^{d+1}$. The group possesses $d+1$ generators defined for $i = 1, \dots, d$, by

$$D_i \boldsymbol{f} = \frac{\partial}{\partial x_i} T_{0,x}\boldsymbol{f}|_{(t,x)=0}, \quad \text{and} \quad D_t \boldsymbol{f} = \frac{\partial}{\partial t} T_{t,0}\boldsymbol{f}|_{(t,x)=0},$$

which are closed and densely defined. Denote by $\mathcal{C}$ the dense subset of $L^2(\Omega)$ defined by

$$\mathcal{C} = \mathrm{Span}\{\boldsymbol{f} * \varphi; \boldsymbol{f} \in L^2(\Omega), \varphi \in C_c^\infty(\mathbb{R}^{d+1})\}, \quad \text{with } \boldsymbol{f} * \varphi(\omega) = \int_{\mathbb{R}^{d+1}} \boldsymbol{f}(\tau_{t,x}\omega)\varphi(t,x)\, dt\, dx,$$

where $C_c^\infty(\mathbb{R}^{d+1})$ is the set of smooth functions on $\mathbb{R}^{d+1}$ with a compact support. Remark that $\mathcal{C} \subset \mathrm{Dom}(D_i)$ and $D_i(\boldsymbol{f} * \varphi) = -\boldsymbol{f} * \frac{\partial \varphi}{\partial x_i}$. This last quantity is also equal to $D_i \boldsymbol{f} * \varphi$ if $\boldsymbol{f} \in \mathrm{Dom}(D_i)$.

Consider now the measurable functions $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d\times d}$, $\widetilde{\boldsymbol{\sigma}} : \Omega \to \mathbb{R}^{d\times d}$, $\boldsymbol{H} : \Omega \to \mathbb{R}^{d\times d}$ and $\boldsymbol{V} : \Omega \to \mathbb{R}$ and assume that $\boldsymbol{H}$ is antisymmetric. Define $\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^*$ and $\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{\sigma}}\widetilde{\boldsymbol{\sigma}}^*$. The function $\boldsymbol{V}$ does not depend on time, that means $\forall t \in \mathbb{R}$, $T_{t,0}\boldsymbol{V} = \boldsymbol{V}$.

Assumption 2.2. (Regularity of the coefficients)
- *Assume that $\forall i,j,k,l = 1,\dots,d$, $\boldsymbol{a}_{ij}, \widetilde{\boldsymbol{a}}_{ij}, \boldsymbol{V}, \boldsymbol{H}_{ij}, D_l\boldsymbol{a}_{ij}$ and $D_l\widetilde{\boldsymbol{a}}_{ij} \in \mathrm{Dom}(D_k)$.*
- *Define, for $i = 1,\dots,d$,*

$$\text{(5)} \quad \begin{aligned} \boldsymbol{b}_i(\omega) &= \sum_{j=1}^d \big(\frac{1}{2} D_j \boldsymbol{a}_{ij}(\omega) - \boldsymbol{a}_{ij} D_j \boldsymbol{V}(\omega) + \frac{1}{2} D_j \boldsymbol{H}_{ij}(\omega)\big), \\ \widetilde{\boldsymbol{b}}_i(\omega) &= \sum_{j=1}^d \big(\frac{1}{2} D_j \widetilde{\boldsymbol{a}}_{ij}(\omega) - \widetilde{\boldsymbol{a}}_{ij} D_j \boldsymbol{V}(\omega)\big), \end{aligned}$$

and assume that the applications $(t,x) \mapsto b_i(t,x,\omega)$, $(t,x) \mapsto \widetilde{b}_i(t,x,\omega)$, $(t,x) \mapsto \sigma(t,x,\omega)$ are globally Lipschitz. Moreover, the coefficients $\boldsymbol{\sigma}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\widetilde{\boldsymbol{\sigma}}$, $\boldsymbol{V}$, $\boldsymbol{H}$ are uniformly bounded by a constant K. (In particular, this ensures existence and uniqueness of a global solution of SDE (8)*.)*

Here is the main assumption of this paper

Assumption 2.3. (Control of the coefficients)
• *$\widetilde{\boldsymbol{\sigma}}$ does not depend on time (i.e. $\forall t \in \mathbb{R}, T_t\widetilde{\boldsymbol{\sigma}} = \widetilde{\boldsymbol{\sigma}}$) and $\boldsymbol{H}, \boldsymbol{a} \in \mathrm{Dom}(D_t)$. As a consequence, the matrix $\widetilde{\boldsymbol{a}}$ does not depend on time either.*
• *There exist five positive constants $m, M, C_1^H, C_2^H, C_2^a$ such that, μ a.s.,*

$$m\widetilde{\boldsymbol{a}} \leq \boldsymbol{a} \leq M\widetilde{\boldsymbol{a}}, \tag{6}$$

$$|\boldsymbol{H}| \leq C_1^H \widetilde{\boldsymbol{a}}, \quad |D_t\boldsymbol{H}| \leq C_2^H \widetilde{\boldsymbol{a}} \quad \textit{and} \quad |D_t\boldsymbol{a}| \leq C_2^a \widetilde{\boldsymbol{a}}, \tag{7}$$

where $|\boldsymbol{A}|$ stands for the symmetric positive square root of $\boldsymbol{A}$, i.e. $|\boldsymbol{A}| = \sqrt{-\boldsymbol{A}^2}$.

For instance, if the matrix $\boldsymbol{a}$ is uniformly elliptic and bounded, $\widetilde{\boldsymbol{\sigma}}$ can be chosen as equal to the identity matrix $\mathbf{Id}$ and then (7) $\Leftrightarrow$ $\boldsymbol{H}$, $D_t\boldsymbol{H}$ and $D_t\boldsymbol{a} \in L^\infty(\Omega)$.
Let us now set out the ergodic properties of this framework

Assumption 2.4. (Ergodicity) *Let us consider the operator $\widetilde{\boldsymbol{S}} = (1/2)e^{2\boldsymbol{V}} \sum_{i,j=1}^d D_i(e^{-2\boldsymbol{V}}\widetilde{\boldsymbol{a}}_{ij}D_j)$ with domain $\mathcal{C}$. From Assumption 2.2, we can consider its Friedrich extension (see [5, Ch. 3, Sect. 3]) which is still denoted $\widetilde{\boldsymbol{S}}$. Assume that each function $\boldsymbol{f} \in \mathrm{Dom}(\widetilde{\boldsymbol{S}})$ satisfying $\widetilde{\boldsymbol{S}}\boldsymbol{f} = 0$ must be μ almost surely equal to some function that is invariant under space translations.*

Even if it means adding to $\boldsymbol{V}$ a constant (and this does not change the drift $\boldsymbol{b}$, see (5)), we make the assumption that $\int e^{-2\boldsymbol{V}}\, d\mu = 1$. Thus we can define a new probability measure on Ω by

$$d\pi(\omega) = e^{-2\boldsymbol{V}(\omega)}\, d\mu(\omega).$$

We now consider a standard d-dimensional Brownian motion defined on a probability space $(\Omega', \mathcal{F}, \mathbb{P})$ (the medium and the Brownian motion are mutually independent) and the diffusions in random medium given as the solutions of the following Stochastic Differential Equations (SDE)

$$\begin{aligned} X_t^\omega &= x + \int_0^t b\left(r, X_r^\omega, \omega\right)\, dr + \int_0^t \sigma\left(r, X_r^\omega, \omega\right)\, dB_r, \\ \widetilde{X}_t^\omega &= x + \int_0^t \widetilde{b}(X_r^\omega, \omega)\, dr + \int_0^t \widetilde{\sigma}(X_r^\omega, \omega)\, dB_r. \end{aligned} \tag{8}$$

The main result of this paper is stated as follows

Theorem 2.5. *The law of the rescaled process $\varepsilon X_{t/\varepsilon^2}^\omega$ converges in probability (with respect to ω) to the law of a Brownian motion with a certain covariance matrix A (see* (45)*).*

3 Examples

There are many ways to ensure the validity of Assumption (2.4). In particular, it is satisfied when, for almost all $\omega \in \Omega$, the $\mathbb{R}^d$-valued Markov process $\widetilde{X}^\omega$, whose generator coincides on smooth functions with

$$\widetilde{\boldsymbol{S}}^\omega = \frac{e^{2V(x,\omega)}}{2}\mathrm{Div}_x\left(e^{-2V(x,\omega)}\widetilde{a}(x,\omega)\nabla_x\,\cdot\right),$$

is irreducible in the following sense. Suppose that, starting from any point of $\mathbb{R}^d$, the process reaches each subset of $\mathbb{R}^d$ of non-null Lebesgue measure in finite time. That means that there exists a measurable subset $N \subset \Omega$ with $\mu(N) = 0$ such that $\forall \omega \in \Omega \setminus N$, for each measurable subset B of $\mathbb{R}^d$ with $\lambda_{Leb}(B) > 0$, $\forall x \in \mathbb{R}^d$, $\exists t > 0$,

$$\mathbb{P}_x\left(\widetilde{X}^\omega_t \in B\right) > 0. \tag{9}$$

This can be proved as in [11] section 3 or in [14] chapter 2 Theorem 2.1, in studying the Ω-valued Markov process $\widetilde{Y}_t(\omega) = \tau_{0,\widetilde{X}^\omega_t}\omega$, whose generator coincides on $\mathcal{C}$ with $\widetilde{\boldsymbol{S}}$. As an easy consequence, if the diffusion coefficient $\widetilde{\boldsymbol{a}}$ is uniformly elliptic or satisfies a strong Hörmander condition (see [9] for further details), then estimates on the transition densities of the process $\widetilde{X}^\omega$ ensure (9).

Let us now tackle the issue of constructing examples that do not satisfy any uniform ellipticity assumption or even strong Hörmander condition. In what follows, two examples are given. The first one deals with periodic coefficients. The second one is a random medium with a random chessboard structure and thereby does not reduce to the periodic case.

3.1 A periodic example

Let us construct a periodic example on the torus $\mathbb{T}^3$, where the diffusion matrix reduces to zero on a certain subset with null Lebesgue measure. We define a time-independent matrix-valued function

$$\widetilde{\boldsymbol{\sigma}}(t,x,y) = (1-\cos(x))(1-\cos(y))\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For simplicity, we choose $\boldsymbol{V} = \boldsymbol{H}(x,y) = 0$. Thanks to the (not uniform!) ellipticity of the diffusion coefficient inside the cell $\mathtt{C} =]0,2\pi[\times]0,2\pi[$, it is not very difficult to see that (9) is satisfied. Indeed, each subset $B \subset [0;2\pi]^2$ with a strictly positive Lebesgue measure necessarily satisfies $\lambda_{Leb}(B \cap \mathtt{C}) > 0$. As explained above, this is sufficient to ensure Assumption 2.4. Let us now focus on Assumption 2.3. The strategy consists in choosing a smooth function $\boldsymbol{U} : \mathbb{T}^3 \to \mathbb{R}^{2\times 2}$ satisfying $\alpha^{-1}\mathrm{Id} \leq \boldsymbol{U}\boldsymbol{U}^*(t,x,y) \leq \alpha\mathrm{Id}$ for some $\alpha > 0$, and then in defining $\boldsymbol{\sigma}(t,x,y) = \widetilde{\boldsymbol{\sigma}}(t,x,y)\boldsymbol{U}(t,x,y)$, for which Assumption 2.3 is easily checked.

3.2 An example on chessboard structures

Let us now explain how to construct a random medium with chessboard structures. Given $d \geq 1$, consider a sequence $(\varepsilon_{(k_1,\dots,k_d)})_{(k_1,\dots,k_d)\in\mathbb{Z}^d}$ of independant Bernouilli random variables with parameter $p \in]0,1[$ and define a process $\widetilde{\eta}$ as follows: for each $x \in \mathbb{R}^d$, there exists a unique $(k_1,\dots,k_d) \in \mathbb{Z}^d$ such that x belongs to the cube $[k_1,k_1+1[\times\cdots\times[k_d,k_d+1[$. Then define the process $\widetilde{\eta} : \mathbb{R}^d \to \mathbb{R}$ by: $\forall x \in \mathbb{R}^d$, $\quad \widetilde{\eta}_x = \varepsilon_{(k_1,\dots,k_d)}$. The law of this process is invariant and ergodic with respect to $\mathbb{Z}^d$ translations. Roughly speaking, we are drawing a d-dimensional chessboard on $\mathbb{R}^d$, for which we are coloring each cube of the chessboard either in black with probability p or in white with probability $1-p$. It remains to make the process invariant under $\mathbb{R}^d$ translations. To this purpose, choose a uniform variable U on the cube $[0,1[^d$ independent of the sequence $(\varepsilon_{(k_1,\dots,k_d)})_{(k_1,\dots,k_d)\in\mathbb{Z}^d}$ and define for $x\in\mathbb{R}^d$, $\overline{\eta}_x=\widetilde{\eta}_{x+U}$. In a way, this corresponds to a random change of the origin of the chessboard. It can be checked that we get a stationary ergodic random field on $\mathbb{R}^d$. Let us now tackle the issue of the regularity of the trajectories. Consider a $C^\infty(\mathbb{R}^d)$ function φ with a compact and very small support (for instance, included in the ball $B(0,1/4)$) and define a new process $\eta_x=\int_{\mathbb{R}^d}\overline{\eta}_y\varphi(x-y)\,dy=\overline{\eta}*\varphi(x)$, which is a stationary ergodic random process with smooth trajectories. That is enough for a general framework.

Let us now consider the process $\omega_{(t,x)}=(\beta_t,\alpha^1_{x_1},\alpha^2_{x_2})_{t\in\mathbb{R},x=(x_1,x_2)\in\mathbb{R}^2}$, where the three processes α^1,α^2 and β are mutually independent and constructed as prescribed above. Hence $\{\omega_{(t,x)};(t,x)\in\mathbb{R}\times\mathbb{R}^2\}$ is an ergodic stationary process and we can consider the random medium $\Omega=C(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^3)$ equipped with the probability law of this process.
We define the matrix $\widetilde{\boldsymbol{\sigma}}(\omega)=\begin{bmatrix}1&0\\0&\alpha^1_0\end{bmatrix}$ and $\boldsymbol{V}=0$ (or any bounded function of the random field α). We can choose any matrix-valued function $\boldsymbol{U}:\Omega\to\mathbb{R}^{2\times2}$ such that $\boldsymbol{U}\boldsymbol{U}^*$ is uniformly elliptic and bounded, and then we set $\boldsymbol{\sigma}=\widetilde{\boldsymbol{\sigma}}\boldsymbol{U}$. It can be proved that Assumption 2.4 is satisfied. Actually, the ergodicity property for $\tilde{\boldsymbol{\sigma}}$ is very intuitive. Indeed, the matrix $\tilde{\sigma}(\cdot,\omega)$ degenerates only on some stripes (the white ones), and in fact only on a part of each of them (depending on the support of φ), and only along the y_2-axis direction: while lying on the degenerating part of a white stripe, the diffusion associated to $(1/2)\sum_{i,j=1}^2\partial_i(\tilde{a}_{i,j}\partial_j)$ can only move along the y_1-axis direction. Nevertheless, with probability 1, the process encounters a black stripe sooner or later (because the parameter p belongs to $]0,1[$): it thus manages to move up and down and to reach every subset of the space. Ergodicity follows. Rigorous arguments are however left to the reader.

We can also consider a non-null stream matrix $\boldsymbol{H}$. For instance the matrix-valued function $\boldsymbol{H}(\omega)=\begin{bmatrix}0&(\alpha^1_0)^2\beta_0\\-(\alpha^1_0)^2\beta_0&0\end{bmatrix}$, fits Assumption 2.3.

4 Environment as seen from the particle

We now look at the *environments as seen from the particle* associated to the processes X and $\widetilde{X}$: they both are Ω-valued Markov processes and are defined by

(10) $$\widetilde{Y}_t(\omega)=\tau_{t,\widetilde{X}^\omega_t}\omega,\quad\text{and}\quad Y_t(\omega)=\tau_{t,X^\omega_t}\omega,$$

where the processes X^ω and $\widetilde{X}^\omega$ both starts from the point $0\in\mathbb{R}^d$. An easy computation proves that the generators of these Markov processes respectively coincide on $\mathcal{C}$ with $\widetilde{\boldsymbol{S}}+D_t$ and $\boldsymbol{L}+D_t$, where $\boldsymbol{L}$ is defined on $\mathcal{C}$ by

(11) $$\boldsymbol{L}=\frac{e^{2\boldsymbol{V}}}{2}\sum_{i,j=1}^d D_i\big(e^{-2\boldsymbol{V}}[\boldsymbol{a}+\boldsymbol{H}]_{ij}D_j\big).$$

Hence π is an invariant measure for both processes (see also [13]). Both associated semigroups thus extend continuously to $L^2(\Omega,\pi)$. We should point out that the invariant measure need not be unique.

5 Poisson's equation

The aim of this section is, at first, to find a solution $\boldsymbol{u}_\lambda$ of the resolvent equation that can formally be rewritten (a rigorous definition of each term is given later), for $\lambda > 0$, as:

$$\lambda \boldsymbol{u}_\lambda - (\boldsymbol{L} + D_t)\boldsymbol{u}_\lambda = \boldsymbol{h}. \tag{12}$$

Since the associated Dirichlet form satisfies no sector condition (even weak), existence and regularity of such a solution is generally a tricky work, especially in considering degeneracies both in time and in space. However, for a suitable right-hand side, this equation can be solved with the help of an approximating sequence of Dirichlet forms satisfying a weak sector condition. Thereafter we study the asymptotic behaviour of the solution $\boldsymbol{u}_\lambda$ as $\lambda \to 0$.

5.1 Setup

Let us denote by $(\widetilde{P}_t)_t$ the semigroup on $L^2(\Omega, \pi)$ generated by the process $\widetilde{Y}$ and by $(\widetilde{P}_t^*)_t$ its adjoint operator. Let us also denote by $(\bar{P}_t)_t$ the self-adjoint semigroup on $L^2(\Omega, \pi)$ generated by the process $\bar{Y}_t(\omega) = \tau_{0,\widetilde{X}_t^\omega}\omega$. Its generator is $\widetilde{\boldsymbol{S}}$. From the time independence of the coefficients $\widetilde{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{\sigma}}$, it is readily seen that, that $\forall \boldsymbol{f} \in L^2(\Omega, \pi)$, $\widetilde{P}_t \boldsymbol{f} = T_{t,0}\bar{P}_t \boldsymbol{f} = \bar{P}_t T_{t,0}\boldsymbol{f}$. As a consequence, $\widetilde{P}_t^* = T_{-t,0}\bar{P}_t \boldsymbol{f} = \bar{P}_t T_{-t,0}\boldsymbol{f}$, in such a way that

$$\widetilde{P}_t(\widetilde{P}_t^* \boldsymbol{f}) = \widetilde{P}_t^*(\widetilde{P}_t \boldsymbol{f}).$$

The generator in $L^2(\Omega, \pi)$ of $(\widetilde{P}_t)_t$, wrongly denoted by $[\widetilde{\boldsymbol{S}} + D_t]$, is then normal (see Theorem 13.38 in [19]) so that we can find a spectral resolution of the identity E on the Borelian subsets of $\mathbb{R}_+ \times \mathbb{R}$ such that

$$-[\widetilde{\boldsymbol{S}} + D_t] = \int_{\mathbb{R}_+ \times \mathbb{R}} (x + iy)\, E(dx, dy).$$

Actually, we have $-\widetilde{\boldsymbol{S}} = \int_{\mathbb{R}_+ \times \mathbb{R}} x\, E(dx, dy)$, and $- D_t = \int_{\mathbb{R}_+ \times \mathbb{R}} iy\, E(dx, dy)$. Indeed, $\widetilde{\boldsymbol{S}}$ and $\int_{\mathbb{R}_+ \times \mathbb{R}} x\, E(dx, dy)$ are both self-adjoint and coincide on $\mathcal{C}$. From [5, Ch. 1, Sect. 3], they are equal. The same arguments hold for D_t and $\int_{\mathbb{R}_+ \times \mathbb{R}} iy\, E(dx, dy)$.

For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in L^2(\Omega)$, denote by $E_{\boldsymbol{\varphi},\boldsymbol{\psi}}$ the measure defined by $E_{\boldsymbol{\varphi},\boldsymbol{\psi}} = (E\boldsymbol{\varphi}, \boldsymbol{\psi})_2$. From now on, denote by $(.\,,.\,)_2$ the usual inner product in $L^2(\Omega, \pi)$. For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, define

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_1 = \int_{\mathbb{R}_+ \times \mathbb{R}} x\, E_{\boldsymbol{\varphi},\boldsymbol{\psi}}(dx, dy) = -(\boldsymbol{\varphi}, \widetilde{\boldsymbol{S}}\boldsymbol{\psi})_2 \tag{13}$$

and $\|\boldsymbol{\varphi}\|_1 = \sqrt{\langle \boldsymbol{\varphi}, \boldsymbol{\varphi} \rangle_1}$. By virtue of Assumption (6), this semi-norm is equivalent on $\mathcal{C}$ to the semi-norm defined by $\sqrt{-(\boldsymbol{\varphi}, \boldsymbol{S}\boldsymbol{\varphi})_2}$,

$$m\|\boldsymbol{\varphi}\|_1^2 \le -(\boldsymbol{\varphi}, \boldsymbol{S}\boldsymbol{\varphi})_2 \le M\|\boldsymbol{\varphi}\|_1^2, \tag{14}$$

where $\boldsymbol{S}$ is the Friedrich extension of the operator defined on $\mathcal{C}$ by $(1/2)e^{2\boldsymbol{V}} \sum_{i,j} D_i\big(e^{-2\boldsymbol{V}}\boldsymbol{a}_{ij} D_j\big)$.

Let $\mathbb{F}$ (respectively $\mathbb{H}$) be the Hilbert space that is the closure of $\mathcal{C}$ in $L^2(\Omega)$ with respect to the inner product ε (resp. κ) defined on $\mathcal{C}$ by

$$\varepsilon(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\boldsymbol{\varphi}, \boldsymbol{\psi})_2 + \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_1 + (D_t\boldsymbol{\varphi}, D_t\boldsymbol{\psi})_2$$
$$(\text{resp. } \kappa(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\boldsymbol{\varphi}, \boldsymbol{\psi})_2 + \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_1).$$

Define the space $\mathbb{D}$ as the closure in $(L^2(\Omega), |\,.\,|_2)$ of the subspace $\{(-\widetilde{S})^{1/2}\boldsymbol{\varphi}; \boldsymbol{\varphi} \in \mathcal{C}\}$. For any $\boldsymbol{\varphi} \in \mathcal{C}$, define $\Phi\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}\big) = \boldsymbol{\sigma}^* D_x\boldsymbol{\varphi} \in (L^2(\Omega))^d$ and note that $|\Phi\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}\big)|_2^2 = -(\boldsymbol{\varphi}, \boldsymbol{S}\boldsymbol{\varphi})_2$. From (14), Φ can be extended to the whole space $\mathbb{D}$ and this extension is a linear isomorphism from $\mathbb{D}$ into a closed subset of $(L^2(\Omega))^d$. Hence, for each function $\boldsymbol{u} \in \mathbb{H}$, we define $\nabla^\sigma \boldsymbol{u} = \Phi((-\widetilde{S})^{1/2}\boldsymbol{u})$ and this stands, in a way, for the gradient of $\boldsymbol{u}$ along the direction σ.
For each $\boldsymbol{f} \in L^2(\Omega)$ satisfying $\int_{\mathbb{R}_+\times\mathbb{R}} \frac{1}{x} E_{\boldsymbol{f},\boldsymbol{f}}(dx, dy) < \infty$, we define

$$\|\boldsymbol{f}\|_{-1}^2 = \int_{\mathbb{R}_+\times\mathbb{R}} \frac{1}{x} E_{\boldsymbol{f},\boldsymbol{f}}(dx, dy). \tag{15}$$

We point out that $\|\boldsymbol{f}\|_{-1} < \infty$ if and only if there exists $C \in \mathbb{R}$ such that for any $\boldsymbol{\varphi} \in \mathcal{C}$, $(\boldsymbol{f}, \boldsymbol{\varphi})_2 \leq C\|\boldsymbol{\varphi}\|_1$. For such a function $\boldsymbol{f}$, $\|\boldsymbol{f}\|_{-1}$ also matches the smallest C satisfying this inequality. Remark that $\|\boldsymbol{f}\|_{-1} < \infty$ implies $\pi(\boldsymbol{f}) = 0$. Denote by $\mathbb{H}_{-1}$ the closure of $L^2(\Omega)$ in $\mathbb{H}^*$ (topological dual of $\mathbb{H}$) with respect to the norm $\|\;\|_{-1}$.
Let us now focus on the antisymmetric part $\boldsymbol{H}$. We have

$$|(u, \boldsymbol{H}v)| \leq (u, |\boldsymbol{H}|u)^{1/2}(v, |\boldsymbol{H}|v)^{1/2} \leq C_1^H (u, \widetilde{\boldsymbol{a}}u)^{1/2}(v, \widetilde{\boldsymbol{a}}v)^{1/2}. \tag{16}$$

The second inequality follows from (7) and the first one is a general fact of linear algebra. We deduce

$$\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}, \quad (1/2)(\boldsymbol{H}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 \leq C_1^H \|\boldsymbol{\psi}\|_1\|\boldsymbol{\varphi}\|_1.$$

Thus there exists an antisymmetric continuous bilinear form $\boldsymbol{T}_H$ on $\mathbb{D} \times \mathbb{D}$ such that

$$\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}, \quad (1/2)(\boldsymbol{H}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 = \boldsymbol{T}_H\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big). \tag{17}$$

Likewise, with the help of Assumption 2.3, we define the continuous bilinear forms $\boldsymbol{T}_a$, $\partial_t\boldsymbol{T}_a$, $\partial_t\boldsymbol{T}_H$, $\Lambda_s\boldsymbol{T}_a$, $\Lambda_s\boldsymbol{T}_a$ on $\mathbb{D} \times \mathbb{D} \subset L^2(\Omega, \pi) \times L^2(\Omega, \pi)$ as follows: $\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$,

$$\begin{aligned}
(1/2)(\boldsymbol{a}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 &= \boldsymbol{T}_a\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big),\\
(1/2)(D_t\boldsymbol{a}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 &= \partial_t\boldsymbol{T}_a\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big),\\
(1/2)(D_t\boldsymbol{H}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 &= \partial_t\boldsymbol{T}_H\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big),\\
(1/2)(\Lambda_s\boldsymbol{a}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 &= \Lambda_s\boldsymbol{T}_a\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big),\\
(1/2)(\Lambda_s\boldsymbol{H}D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 &= \Lambda_s\boldsymbol{T}_H\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big),
\end{aligned}$$

where, for any $s \in \mathbb{R}^*$, Λ_s denotes the L^2-continuous difference operator (remind of the definition of $T_{s,0}$ in section 2):

$$\forall \boldsymbol{f} \in L^2(\Omega), \quad \Lambda_s(\boldsymbol{f}) = (T_{s,0}\boldsymbol{f} - \boldsymbol{f})/s. \tag{18}$$

From Assumption 2.3, the norms of the forms $\Lambda_s\boldsymbol{T}_a$ and $\Lambda_s\boldsymbol{T}_H$ are uniformly bounded with respect to $s \in \mathbb{R}^*$ and the forms are weakly convergent respectively towards $\partial_t\boldsymbol{T}_a$ and $\partial_t\boldsymbol{T}_H$.
Now, denote by $\mathcal{H}$ the subspace of $\mathbb{H}_{-1}$ whose elements satisfy the condition: $\exists C > 0, \forall s > 0$ and $\forall \boldsymbol{\varphi} \in \mathcal{C}$, $\langle \boldsymbol{h}, \Lambda_s\boldsymbol{\varphi}\rangle_{-1,1} \leq C\|\boldsymbol{\varphi}\|_1$. For any $\boldsymbol{h} \in \mathcal{H}$, the smallest C that satisfies such a condition is denoted $\|\boldsymbol{h}\|_T$. Then $\mathcal{H}$ is closed for the norm $\|\;\|_{\mathcal{H}} = \|\;\|_{-1} + \|\;\|_T$.
Finally, let us now extend the operator $\boldsymbol{L}$ defined on $\mathcal{C}$ by (11). For any $\lambda > 0$, consider the continuous bilinear form $\mathcal{B}_\lambda$ on $\mathbb{H} \times \mathbb{H}$ that coincides on $\mathcal{C} \times \mathcal{C}$ with

$$\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}, \quad \mathcal{B}_\lambda(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \lambda(\boldsymbol{\varphi}, \boldsymbol{\psi})_2 + [\boldsymbol{T}_a + \boldsymbol{T}_H]\big((-\widetilde{S})^{1/2}\boldsymbol{\varphi}, (-\widetilde{S})^{1/2}\boldsymbol{\psi}\big).$$

Thanks to Assumption 2.3 and the antisymmetry of $\boldsymbol{H}$, this form is clearly coercive. Thus it defines a strongly continuous resolvent operator and consequently, the generator $\boldsymbol{L}$ associated to this resolvent operator. More precisely, $\boldsymbol{\varphi} \in \mathbb{H}$ belongs to $\mathrm{Dom}(\boldsymbol{L})$ if and only if $\mathcal{B}_\lambda(\boldsymbol{\varphi}, \cdot)$ is L^2-continuous. In this case, there exists $\boldsymbol{f} \in L^2(\Omega)$ such that $\mathcal{B}_\lambda(\boldsymbol{\varphi}, \cdot) = (\boldsymbol{f}, \cdot)_2$ and $\boldsymbol{L}\boldsymbol{\varphi}$ is equal to $\boldsymbol{f} - \lambda\boldsymbol{\varphi}$. It can be proved that this definition is independent of $\lambda > 0$ (see [12, Ch. 1, Sect. 2] for further details). Let us additionally mention that the adjoint operator $\boldsymbol{L}^*$ of $\boldsymbol{L}$ in $L^2(\Omega, \pi)$ can also be described through $\mathcal{B}_\lambda$. Indeed, $\mathrm{Dom}(\boldsymbol{L}^*) = \{\boldsymbol{\varphi} \in \mathbb{H}; \mathcal{B}_\lambda(\cdot, \boldsymbol{\varphi})$ is $L^2(\Omega)$-continuous.$\}$. If $\boldsymbol{\varphi} \in \mathrm{Dom}(\boldsymbol{L}^*)$, there exists $\boldsymbol{f} \in L^2(\Omega)$ such that $\mathcal{B}_\lambda(\cdot, \boldsymbol{\varphi}) = (\boldsymbol{f}, \cdot)_2$ and $\boldsymbol{L}^*\boldsymbol{\varphi}$ is equal to $\boldsymbol{f} - \lambda\boldsymbol{\varphi}$.

Remark 5.2. *For each function $\boldsymbol{\varphi} \in \mathcal{C} \subset \mathbb{H}$, the application $\boldsymbol{L}\boldsymbol{\varphi}$ can be viewed as a function of $\mathbb{H}_{-1}$. Indeed, $\forall \boldsymbol{\psi} \in \mathcal{C}$, $(\boldsymbol{L}\boldsymbol{\varphi}, \boldsymbol{\psi})_2 = -[\boldsymbol{T}_a + \boldsymbol{T}_H]\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\psi}\big) \leq [M + C_1^H]\|\boldsymbol{\varphi}\|_1\|\boldsymbol{\psi}\|_1$. Hence, the application $\boldsymbol{\varphi} \mapsto \boldsymbol{L}\boldsymbol{\varphi} \in \mathbb{H}_{-1}$ can be extended to the whole space $\mathbb{H}$ so that, for each function $\boldsymbol{u} \in \mathbb{H}$, we can define $\boldsymbol{L}\boldsymbol{u}$ as an element of $\mathbb{H}_{-1}$ even if $\boldsymbol{u} \notin \mathrm{Dom}(\boldsymbol{L})$.*

5.3 Existence of a solution:

This section is devoted to proving existence of solutions of equation (12) for a suitable right-hand side. The difficulty lies in the strong degeneracy of the associated Dirichlet form. It satisfies no sector condition, even weak. However, it can be approximated by a family of Dirichlet forms with weak sector condition.
For any $\theta \in \{0; 1\}$, $\lambda > 0$ and $\delta \geq 0$, define $B^\theta_{\lambda,\delta}$ as the (non-symmetric) bilinear continuous form on $\mathbb{F} \times \mathbb{F}$ that coincides on $\mathcal{C} \times \mathcal{C}$ with

(19) $$B^\theta_{\lambda,\delta}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \lambda(\boldsymbol{\varphi}, \boldsymbol{\psi})_2 + (1/2)([\boldsymbol{a} + \boldsymbol{H}]D_x\boldsymbol{\varphi}, D_x\boldsymbol{\psi})_2 - \theta(D_t\boldsymbol{\varphi}, \boldsymbol{\psi})_2 + (\delta/2)(D_t\boldsymbol{\varphi}, D_t\boldsymbol{\psi})_2.$$

In what follows, the parameter θ (resp. δ) is omitted each time that it is equal to 1 (resp. 0). So the forms $B^1_{\lambda,\delta}$, $B^\theta_{\lambda,0}$ and $B^1_{\lambda,0}$ are respectively simply denoted by $B_{\lambda,\delta}$, B^θ_λ and B_λ.

Proposition 5.4. *Suppose that $\boldsymbol{h} \in L^2(\Omega) \cap \mathrm{Dom}(D_t)$ and $\boldsymbol{d} \in \mathcal{H}$. Then, for any $\theta \in \{0; 1\}$ and $\lambda > 0$, there exists a unique solution $\boldsymbol{u}_\lambda \in \mathbb{F}$ of the equation $\lambda\boldsymbol{u}_\lambda - \boldsymbol{L}\boldsymbol{u}_\lambda - \theta D_t\boldsymbol{u}_\lambda = \boldsymbol{h} + \boldsymbol{d}$, in the sense that $\forall \boldsymbol{\varphi} \in \mathbb{F}$, $B^\theta_\lambda(\boldsymbol{u}_\lambda, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2 + \langle \boldsymbol{d}, \boldsymbol{\varphi}\rangle_{-1,1}$. Moreover, $D_t\boldsymbol{u}_\lambda \in \mathbb{H}$ and*

(20a) $$\lambda|\boldsymbol{u}_\lambda|_2^2 + m\|\boldsymbol{u}_\lambda\|_1^2 \leq |\boldsymbol{h}|_2^2/\lambda + \|\boldsymbol{d}|_{-1}^2/m,$$

(20b) $$\lambda|D_t\boldsymbol{u}_\lambda|_2^2 + m\|D_t\boldsymbol{u}_\lambda\|_1^2 \leq |D_t\boldsymbol{h}|_2^2/\lambda + 2\|\boldsymbol{d}\|_T^2/m + 2(C_2^a + C_2^H)^2\big(|\boldsymbol{h}|^2/\lambda + \|\boldsymbol{d}\|_{-1}^2/m\big)/m^2.$$

In the case $\boldsymbol{d} \in L^2(\Omega)$, $\boldsymbol{u}_\lambda \in \mathrm{Dom}(\boldsymbol{L})$.
Finally, $\boldsymbol{u}_\lambda$ is the strong limit in $\mathbb{H}$ as δ goes to 0 of the sequence $(\boldsymbol{u}_{\lambda,\delta})_{\lambda,\delta}$, where $\boldsymbol{u}_{\lambda,\delta}$ is the unique solution of the equation: $\forall \boldsymbol{\varphi} \in \mathbb{F}$, $B^\theta_{\lambda,\delta}(\boldsymbol{u}_{\lambda,\delta}, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2 + \langle \boldsymbol{d}, \boldsymbol{\varphi}\rangle_{-1,1}$, and the family $(D_t\boldsymbol{u}_{\lambda,\delta})_\delta$ is bounded in $L^2(\Omega)$.

Before proving this result, we first investigate the case of time independent coefficients. On the first side, this is a good starting point for understanding the proof in the time dependent case and this will bring out the difficulties arising with the time dependency. On the other side, this result is needed in the last section of this paper in order to prove the tightness of the process X.

Proposition 5.5. *Suppose that $\boldsymbol{h} \in L^2(\Omega)$ Then, for any $\lambda > 0$, there exists a unique solution $\boldsymbol{w}_\lambda \in \mathbb{H} \cap \mathrm{Dom}(\boldsymbol{S})$ of the equation*

(21) $$\lambda\boldsymbol{w}_\lambda - \boldsymbol{S}\boldsymbol{w}_\lambda = \boldsymbol{h}.$$

Proof : The main tool of this proof is the Lax-Milgram theorem. Let $\lambda > 0$ be fixed. For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, consider the bilinear form on $\mathcal{C} \times \mathcal{C}$ defined by

$$D_\lambda(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \lambda(\boldsymbol{\varphi}, \boldsymbol{\psi})_2 - (\boldsymbol{\varphi}, \boldsymbol{S}\boldsymbol{\psi})_2.$$

Thanks to Assumption 2.3, this form is clearly coercive and continuous on $\mathcal{C} \times \mathcal{C}$ so that it can be extended to the whole space $\mathbb{H} \times \mathbb{H}$. The extension is also coercive and continuous. Now, the application $\boldsymbol{\varphi} \mapsto (\boldsymbol{h}, \boldsymbol{\varphi})_2$ is obviously continuous on $\mathbb{H}$ so that the Lax-Milgram theorem applies. It allows to construct a strongly continuous resolvent associated to $\lambda - \boldsymbol{S}$ by way of classical tools (see [5, Ch. 1, Sect. 3] or [12, Ch. 1, Sect. 2] for further details). □

Proof of the Proposition 5.4: Since the case $\theta = 0$ and $\theta = 1$ are quite similar, we only give the proof for $\theta = 1$. The existence of a solution relies on the Lax-Milgram theorem again. However, the considered bilinear form (19) with $\delta = 0$ is not coercive on $\mathbb{F}$ because of the time differential term $(D_t\boldsymbol{\varphi}, \boldsymbol{\psi})$. The strategy consists in making it coercive by adding a term $(\delta/2)(D_t\boldsymbol{\varphi}, D_t\boldsymbol{\psi})$ ($\delta > 0$) and then letting δ go to 0. Notice that for $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, we have

$$\big([\lambda - \boldsymbol{L} - D_t - (\delta/2)D_t^2]\boldsymbol{\varphi}, \boldsymbol{\psi}\big)_2 = B_{\lambda,\delta}(\boldsymbol{\varphi}, \boldsymbol{\psi}).$$

The continuity of $B_{\lambda,\delta}$ on $\mathcal{C} \times \mathcal{C} \subset \mathbb{F} \times \mathbb{F}$ follows from (6) and (16). As a result of the time-independence of $\boldsymbol{V}$, for any $\boldsymbol{\varphi} \in \mathcal{C}$, we have $(\boldsymbol{\varphi}, D_t\boldsymbol{\varphi})_2 = 0$. As a consequence, for any $\boldsymbol{\varphi} \in \mathcal{C}$,

$$\min(\lambda, \delta/2, m)\varepsilon(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \leq B_{\lambda,\delta}(\boldsymbol{\varphi}, \boldsymbol{\varphi}). \tag{22}$$

Hence $B_{\lambda,\delta}$ defines a continuous coercive bilinear form on $\mathbb{F}\times\mathbb{F}$. The Lax-Milgram theorem applies and provides us with a solution $\boldsymbol{u}_{\lambda,\delta}$ of the equation

$$\forall \boldsymbol{\varphi} \in \mathcal{C}, \quad B_{\lambda,\delta}(\boldsymbol{u}_{\lambda,\delta}, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2 + \langle \boldsymbol{d}, \boldsymbol{\varphi}\rangle_{-1,1}. \tag{23}$$

In particular, choosing $\boldsymbol{\varphi} = \boldsymbol{u}_{\lambda,\delta}$ in (23), we get the bound

$$\lambda|\boldsymbol{u}_{\lambda,\delta}|_2^2 + m\|\boldsymbol{u}_{\lambda,\delta}\|_1^2 + \delta|D_t\boldsymbol{u}_{\lambda,\delta}|_2^2 \leq |\boldsymbol{h}|_2^2/\lambda + \|\boldsymbol{d}\|_{-1}^2/m. \tag{24}$$

Let us now to pass to the limit as δ goes to 0 to obtain a solution $\boldsymbol{u}_\lambda \in \mathbb{F}$ of the equation

$$\forall \boldsymbol{\varphi} \in \mathcal{C}, \quad B_\lambda(\boldsymbol{u}_\lambda, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2 + \langle \boldsymbol{d}, \boldsymbol{\varphi}\rangle_{-1,1}. \tag{25}$$

We are faced with the problem of controlling $D_t\boldsymbol{u}_{\lambda,\delta}$ as δ goes to 0. The idea lies in differentiating equation (23) with respect to the time variable in order to establish an equation satisfied by $D_t\boldsymbol{u}_{\lambda,\delta}$, from which estimates will be derived. So, we define for each fixed $\lambda, \delta > 0$, $\boldsymbol{v}_s = \Lambda_s\boldsymbol{u}_{\lambda,\delta}$ (the parameters λ, δ of $\boldsymbol{v}_s$ are temporarily omitted in order to simplify the notations) and we easily check that $\boldsymbol{v}_s$ solves the following equation

$$\forall \boldsymbol{\varphi} \in \mathbb{F},\ B_{\lambda,\delta}(\boldsymbol{v}_s, \boldsymbol{\varphi}) = \boldsymbol{F}_s(\boldsymbol{\varphi}), \tag{26}$$

where $\boldsymbol{F}_s$ is a continuous linear form on $\mathbb{F}$ defined, $\forall \boldsymbol{\varphi} \in \mathbb{F}$, by

$$\boldsymbol{F}_s(\boldsymbol{\varphi}) = (\Lambda_s\boldsymbol{h}, \boldsymbol{\varphi})_2 - \langle \boldsymbol{d}, \Lambda_{-s}\boldsymbol{\varphi}\rangle_{-1,1} - [\Lambda_s\boldsymbol{T}_a + \Lambda_s\boldsymbol{T}_H]\big((-\widetilde{\boldsymbol{S}})^{1/2}T_{s,0}\boldsymbol{u}_{\lambda,\delta}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}\big). \tag{27}$$

From Assumption 2.3, it is readily seen that

$$\boldsymbol{F}_s(\boldsymbol{\varphi}) \leq |D_t\boldsymbol{h}|_2|\boldsymbol{\varphi}|_2 + \|\boldsymbol{d}\|_T\|\boldsymbol{\varphi}\|_1 + (C_2^a + C_2^H)\|\boldsymbol{u}_{\lambda,\delta}\|_1\|\boldsymbol{\varphi}\|_1,$$

for any $s \in \mathbb{R}^*$. Therefore

$$B_{\lambda,\delta}(\boldsymbol{v}_s, \boldsymbol{v}_s) = \boldsymbol{F}_s(\boldsymbol{v}_s) \leq |D_t \boldsymbol{h}|_2 |\boldsymbol{v}_s|_2 + \|\boldsymbol{d}\|_T \|\boldsymbol{\varphi}\|_1 + (C_2^a + C_2^H)\|\boldsymbol{u}_{\lambda,\delta}\|_1 \|\boldsymbol{v}_s\|_1. \tag{28}$$

Using estimate (24) in (28), we have

$$\lambda|\boldsymbol{v}_s|_2^2 + m\|\boldsymbol{v}_s\|_1^2 + \delta|D_t\boldsymbol{v}_s|_2^2 \leq |D_t\boldsymbol{h}|_2^2/\lambda + 2\|\boldsymbol{d}\|_T^2/m + 2(C_2^a + C_2^H)^2\big(|\boldsymbol{h}|^2/\lambda + \|\boldsymbol{d}\|_{-1}^2/m\big)/m^2. \tag{29}$$

So, the family $(\boldsymbol{v}_s)_{s\in\mathbb{R}^*}$ is bounded in $\mathbb{F}$. Even if it means extracting a subsequence (still denoted by $(\boldsymbol{v}_s)_{s\in\mathbb{R}^*}$), $(\boldsymbol{v}_s)_{s\in\mathbb{R}^*}$ converges weakly in $\mathbb{F}$ towards some function $\boldsymbol{v}_0 \in \mathbb{F}$ as s goes to 0. On the other hand, since $\boldsymbol{u}_{\lambda,\delta} \in \mathbb{F} \subset \mathrm{Dom}(D_t)$, $(\boldsymbol{v}_s)_{s\in\mathbb{R}^*}$ also converges strongly in $L^2(\Omega)$ towards $D_t\boldsymbol{u}_{\lambda,\delta}$, so that $D_t\boldsymbol{u}_{\lambda,\delta} \in \mathbb{F}$ and satisfies bound (29) instead of $\boldsymbol{v}_s$. In particular, $(D_t\boldsymbol{u}_{\lambda,\delta})_{\delta>0}$ is bounded in $\mathbb{H}$ independently of $\delta > 0$ and so is $(\boldsymbol{u}_{\lambda,\delta})_{\delta>0}$ in $\mathbb{F}$. Thereby, there exists a subsequence $(\boldsymbol{u}_{\lambda,\delta}, D_t\boldsymbol{u}_{\lambda,\delta})_{\delta>0} \subset \mathbb{F} \times \mathbb{H}$, still indexed with $\delta > 0$, that converges weakly in $\mathbb{F} \times \mathbb{H}$ towards $(\boldsymbol{u}_\lambda, D_t\boldsymbol{u}_\lambda) \in \mathbb{F} \times \mathbb{H}$ as $\delta \to 0$. In particular, $\delta D_t\boldsymbol{u}_{\lambda,\delta} \to 0$ in $L^2(\Omega)$ as δ goes to 0. So we are in position to pass to the limit as δ goes to 0 in (23). Obviously, $\boldsymbol{u}_\lambda$ is a solution of (25). Uniqueness of the weak limit raises no particular difficulty since two weak limits $\boldsymbol{u}_\lambda$ and $\boldsymbol{w}_\lambda$ satisfy $\forall\boldsymbol{\varphi} \in \mathbb{F}$, $B_\lambda(\boldsymbol{u}_\lambda - \boldsymbol{w}_\lambda, \boldsymbol{\varphi}) = 0$. It just remains to choose $\boldsymbol{\varphi} = \boldsymbol{u}_\lambda - \boldsymbol{w}_\lambda$. (20a) and (20b) respectively result from (24) and (29). If $\boldsymbol{d} \in L^2(\Omega)$, note that $\boldsymbol{u}_\lambda \in \mathbb{F} \subset \mathbb{H}$ and $\mathcal{B}_\lambda(\boldsymbol{u}_\lambda, \cdot) = (\boldsymbol{h} + \boldsymbol{d} + D_t\boldsymbol{u}_\lambda, \cdot)_2$ is L^2-continuous so that $\boldsymbol{u}_\lambda \in \mathrm{Dom}(\boldsymbol{L})$.
Let us now investigate the strong convergence in $\mathbb{F}$ of $(\boldsymbol{u}_{\lambda,\delta})_{\lambda,\delta}$ towards $\boldsymbol{u}_\lambda$ as δ goes to 0. Let us make the difference between (23) and (25) and choose $\boldsymbol{\varphi} = \boldsymbol{u}_{\lambda,\delta} - \boldsymbol{u}_\lambda$, this yields

$$B_{\lambda,\delta}(\boldsymbol{u}_{\lambda,\delta} - \boldsymbol{u}_\lambda, \boldsymbol{u}_{\lambda,\delta} - \boldsymbol{u}_\lambda) = (\delta/2)(D_t\boldsymbol{u}_\lambda, D_t\boldsymbol{u}_\lambda - D_t\boldsymbol{u}_{\lambda,\delta})_2,$$

and this latter quantity converges to 0 as δ goes to 0 because of the boundedness of the family $(|D_t\boldsymbol{u}_{\lambda,\delta}|_2)_{\lambda,\delta}$. (22) allows to conclude. □

5.6 Control of the solution

Our goal is now to determine the asymptotic behaviour, as λ goes to 0, of the solution $\boldsymbol{u}_\lambda^i$ of the equation (in the sense of Proposition 5.4)

$$\lambda\boldsymbol{u}_\lambda^i - \boldsymbol{L}\boldsymbol{u}_\lambda^i - D_t\boldsymbol{u}_\lambda^i = \boldsymbol{b}_i. \tag{30}$$

More precisely, we aim at proving that $\lambda|\boldsymbol{u}_\lambda^i|_2^2 \to 0$ and that $(\nabla^\sigma\boldsymbol{u}_\lambda^i)_\lambda$ converges in $(L^2(\Omega))^d$ as λ goes to 0. Our strategy consists in showing that the operator $\lambda - \boldsymbol{L} - D_t$ is just a perturbation of the operator $\lambda - \widetilde{\boldsymbol{S}} - D_t$, so that the study can be reduced to studying the solution of the equation

$$\lambda\boldsymbol{v}_\lambda - \widetilde{\boldsymbol{S}}\boldsymbol{v}_\lambda - D_t\boldsymbol{v}_\lambda = \boldsymbol{b}_\lambda,$$

where $\boldsymbol{b}_\lambda$ will be defined thereafter but possesses a strong limit in $\mathbb{H}_{-1}$. This latter equation is more convenient to study because the operators $\widetilde{\boldsymbol{S}}$ and D_t can be viewed through the same spectral decomposition. Thus, the purpose of this section is to prove the following Proposition

Proposition 5.7. *Let $(\boldsymbol{b}_\lambda)_{\lambda>0}$ be a family of functions in $\mathbb{H}_{-1} \cap L^2(\Omega)$ which is strongly convergent in $\mathbb{H}_{-1}$ to $\boldsymbol{b}_0$. Suppose that there exists a constant C (which does not depend on λ) such that $\forall s > 0$ and $\forall\boldsymbol{\varphi} \in \mathcal{C}$,*

$$(\boldsymbol{b}_\lambda, \Lambda_s\boldsymbol{\varphi})_2 \leq C\|\boldsymbol{\varphi}\|_1.$$

Then the solution $\boldsymbol{u}_\lambda \in \mathbb{F}$ of the equation $\lambda \boldsymbol{u}_\lambda - \boldsymbol{L}\boldsymbol{u}_\lambda - D_t \boldsymbol{u}_\lambda = \boldsymbol{b}_\lambda$ (in the sense of Proposition 5.4) satisfies:

- *there exists $\boldsymbol{\eta} \in \mathbb{D}$ such that $(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}_\lambda \to \boldsymbol{\eta}$ as λ goes to 0 in $\mathbb{D}$,*
- *$\lambda |\boldsymbol{u}_\lambda|_2^2 \to 0$ as λ goes to 0.*

As for the existence of the solution, let us first investigate the time independent case by way of introduction.

Proposition 5.8. *Let $\boldsymbol{h}$ be in $\mathbb{H}_{-1} \cap L^2(\Omega)$. For any $\lambda > 0$, let $\boldsymbol{w}_\lambda$ be defined as the unique solution in $\mathbb{H}$ of the equation*

$$\lambda \boldsymbol{w}_\lambda - \boldsymbol{S}\boldsymbol{w}_\lambda = \boldsymbol{h}$$

Then $\lambda |\boldsymbol{w}_\lambda|_2^2 \to 0$ and there exists $\boldsymbol{\zeta} \in (L^2(\Omega))^d$ such that $|\nabla^\sigma \boldsymbol{w}_\lambda - \boldsymbol{\zeta}|_2 \to 0$ as λ goes to 0.

Proof : Keeping the notations of Proposition 5.5, $\boldsymbol{w}_\lambda$ solves the equation: $\forall \boldsymbol{\varphi} \in \mathbb{H}$, $D_\lambda(\boldsymbol{w}_\lambda, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2$. Choosing $\boldsymbol{\varphi} = \boldsymbol{w}_\lambda$ and using $\boldsymbol{h} \in \mathbb{H}_{-1}$, we have $\lambda |\boldsymbol{w}_\lambda|_2^2 + m\|\boldsymbol{w}_\lambda\|_1^2 \leq \|\boldsymbol{h}\|_{-1}^2/m$. Thus, even if it means extracting a subsequence, we can find $\boldsymbol{g} \in L^2(\Omega)$ such that $((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda)_\lambda$ converges weakly in $L^2(\Omega)$ towards $\boldsymbol{g}$ as λ tends to 0. Moreover $(\lambda \boldsymbol{w}_\lambda)_\lambda$ clearly converges to 0 in $L^2(\Omega)$. For any $\boldsymbol{\varphi} \in \mathbb{H}$, passing to the limit as λ goes to zero in the expression

$$\lambda(\boldsymbol{w}_\lambda, \boldsymbol{\varphi})_2 + \boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi})_2 = D_\lambda(\boldsymbol{w}_\lambda, \boldsymbol{\varphi}) = (\boldsymbol{h}, \boldsymbol{\varphi})_2, \tag{31}$$

we obtain $\boldsymbol{T}_a\big(\boldsymbol{g}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}\big)_2 = (\boldsymbol{h}, \boldsymbol{\varphi})_2$. Making the difference between the last two equalities, subtracting $\boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda - \boldsymbol{g}, \boldsymbol{g}\big)$ and then choosing $(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi} = (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda - \boldsymbol{g}$, we obtain

$$\lambda |\boldsymbol{w}_\lambda|_2^2 + \boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda - \boldsymbol{g}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda - \boldsymbol{g}\big) = -\boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda - \boldsymbol{g}, \boldsymbol{g}\big).$$

Due to the weak convergence of $((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda)_\lambda$ to $\boldsymbol{g}$ in $\mathbb{D}$, the right-hand side converges to 0 as λ goes to 0. So does the left-hand side. Since $\boldsymbol{T}_a$ defines an inner product on $\mathbb{D}$ equivalent to the canonical one (Assumption 2.3), this completes the proof of the strong convergence up to a subsequence. Uniqueness of the weak limit is clear since two weak limits $\boldsymbol{g}$ and $\boldsymbol{g}' \in \mathbb{D}$ satisfy: $\forall \boldsymbol{\varphi} \in \mathcal{C}$, $\boldsymbol{T}_a(\boldsymbol{g} - \boldsymbol{g}', (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}) = 0$. Finally, since the convergence in $\mathbb{D}$ of $((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{w}_\lambda)_\lambda$ is equivalent to the convergence of $(\nabla^\sigma \boldsymbol{w}_\lambda)_\lambda$ in $(L^2(\Omega))^d$, we complete the proof. $\square$

Proposition 5.9. *Let $(\boldsymbol{b}_\lambda)_{\lambda>0}$ be a family of functions in $\mathbb{H}_{-1}$ that is strongly convergent to $\boldsymbol{b}_0$ in $\mathbb{H}_{-1}$. Let $(\boldsymbol{v}_\lambda)_{\lambda>0}$ be a family of functions in $\mathbb{F}$ that solves the equation (for any $\lambda > 0$) $\lambda \boldsymbol{v}_\lambda - \widetilde{\boldsymbol{S}}\boldsymbol{v}_\lambda - D_t \boldsymbol{v}_\lambda = \boldsymbol{b}_\lambda$ in the following sense,*

$$\forall \boldsymbol{\varphi} \in \mathbb{F}, \quad \lambda(\boldsymbol{v}_\lambda, \boldsymbol{\varphi})_2 + \langle \boldsymbol{v}_\lambda, \boldsymbol{\varphi}\rangle_1 - (D_t \boldsymbol{v}_\lambda, \boldsymbol{\varphi})_2 = (\boldsymbol{b}_\lambda, \boldsymbol{\varphi})_2. \tag{32}$$

Then there exists $\boldsymbol{\eta} \in \mathbb{D}$ such that $\lambda |\boldsymbol{v}_\lambda|_2^2 \to 0$ and $|(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda - \boldsymbol{\eta}|_2 \to 0$ as λ goes to 0.

Proof: From Lemma 5.10 and Lemma 5.11 below, we can assume that, for any $\lambda > 0$, $\boldsymbol{b}_\lambda \in L^2(\Omega) \cap \mathrm{Dom}(D_t) \cap \mathbb{H}_{-1}$ and converges to $\boldsymbol{b}_0 \in \mathbb{H}_{-1}$. Then $\boldsymbol{v}_\lambda \in \mathrm{Dom}(\widetilde{\boldsymbol{S}})$ (see Proposition 5.4). Remind that $-\widetilde{\boldsymbol{S}} = \int_{\mathbb{R}_+ \times \mathbb{R}} x\, E(dx, dy)$ and $-D_t = \int_{\mathbb{R}_+ \times \mathbb{R}} iy\, E(dx, dy)$. Choosing $\boldsymbol{\varphi} = \boldsymbol{v}_\lambda$ in (32), we have

$$\lambda |\boldsymbol{v}_\lambda|_2^2 + \|\boldsymbol{v}_\lambda\|_1^2 = (\boldsymbol{b}_\lambda, \boldsymbol{v}_\lambda)_2 \leq C\|\boldsymbol{v}_\lambda\|_1 \leq C^2, \tag{33}$$

where $C = \sup_{\lambda>0} \|\boldsymbol{b}_\lambda\|_{-1}$. Thus we can find $\boldsymbol{h} \in \mathbb{D}$ and a subsequence, still denoted by $(\boldsymbol{v}_\lambda)_\lambda$, such that $\left((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda\right)_\lambda$ converges weakly in $L^2(\Omega)$ to $\boldsymbol{h}$.
Now we claim $\sup_{\lambda>0} \|\lambda \boldsymbol{v}_\lambda\|_{-1} < \infty$ and $\sup_{\lambda>0} \|D_t \boldsymbol{v}_\lambda\|_{-1} < \infty$.

$$
\begin{aligned}
|(\lambda \boldsymbol{v}_\lambda, \boldsymbol{\varphi})_2| &= \Big|\int_{\mathbb{R}_+\times\mathbb{R}} \lambda(\lambda + x + iy)^{-1}\, dE_{\boldsymbol{b}_\lambda,\boldsymbol{\varphi}}\Big| \\
&\leq \Big(\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\lambda^2}{x[(\lambda+x)^2+y^2]}\, dE_{\boldsymbol{b}_\lambda,\boldsymbol{b}_\lambda}\Big)^{1/2}\Big(\int_{\mathbb{R}_+\times\mathbb{R}} x\, dE_{\boldsymbol{\varphi},\boldsymbol{\varphi}}\Big)^{1/2} \\
&\leq \sup_{\lambda>0}\Big(\int_{\mathbb{R}_+\times\mathbb{R}} x^{-1}\, dE_{\boldsymbol{b}_\lambda,\boldsymbol{b}_\lambda}\Big)^{1/2}\|\boldsymbol{\varphi}\|_1 \\
&= \sup_{\lambda>0} \|\boldsymbol{b}_\lambda\|_{-1}\|\boldsymbol{\varphi}\|_1.
\end{aligned}
$$

Since $D_t\boldsymbol{v}_\lambda = \lambda\boldsymbol{v}_\lambda - \widetilde{\boldsymbol{S}}\boldsymbol{v}_\lambda - \boldsymbol{b}_\lambda$ and $\|\widetilde{\boldsymbol{S}}\boldsymbol{v}_\lambda\|_{-1} \leq \|\boldsymbol{v}_\lambda\|_1$, $D_t\boldsymbol{v}_\lambda \in \mathbb{H}_{-1}$ and $\sup_{\lambda>0}\|D_t\boldsymbol{v}_\lambda\|_{-1} < \infty$. Then there exists a bounded family $(\boldsymbol{F}_\lambda)_{\lambda\geq 0}$ of continuous linear forms on $\mathbb{D} \subset L^2(\Omega)$ such that $\forall \lambda > 0, \forall \boldsymbol{\varphi} \in \mathcal{C}, \quad \boldsymbol{F}_\lambda((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}) = (D_t\boldsymbol{v}_\lambda, \boldsymbol{\varphi})_2$. Moreover, from (33), $(\lambda\boldsymbol{v}_\lambda)_\lambda$ converges to 0 in $L^2(\Omega)$ so that, $\forall \boldsymbol{\varphi} \in \mathcal{C}$

$$
\begin{aligned}
\boldsymbol{F}_\lambda((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}) &= (\lambda\boldsymbol{v}_\lambda, \boldsymbol{\varphi})_2 + ((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi})_2 - \langle \boldsymbol{b}_\lambda, \boldsymbol{\varphi}\rangle_{-1,1} \\
&\to (\boldsymbol{h}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi})_2 - \langle \boldsymbol{b}_0, \boldsymbol{\varphi}\rangle_{-1,1}
\end{aligned}
$$

as λ goes to 0. Hence, $(\boldsymbol{F}_\lambda)_{\lambda\geq 0}$ is weakly convergent in $\mathbb{D}^*$ (topological dual of $\mathbb{D}$) to a limit denoted by $\boldsymbol{F}_0$.
We now aim at proving $\boldsymbol{F}_0(\boldsymbol{h}) = 0$. Using the antisymmetry of the operator D_t

$$\boldsymbol{F}_\lambda((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\mu) = (D_t\boldsymbol{v}_\lambda, \boldsymbol{v}_\mu)_2 = -(D_t\boldsymbol{v}_\mu \boldsymbol{v}_\lambda)_2 = -\boldsymbol{F}_\mu((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda),$$

we pass to the limit as λ goes to 0 and obtain $\boldsymbol{F}_0((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\mu) = -\boldsymbol{F}_\mu(\boldsymbol{h})$. It just remains to pass to the limit as μ goes to 0, it yields $\boldsymbol{F}_0(\boldsymbol{h}) = -\boldsymbol{F}_0(\boldsymbol{h}) = 0$.
Let us investigate now the limit equation, which connects $\boldsymbol{F}_0$, $\boldsymbol{h}$ and $\boldsymbol{b}_0$. First remind of (33), which states $\lambda|\boldsymbol{v}_\lambda|_2^2 \leq C^2$ and as a consequence $\lambda\boldsymbol{v}_\lambda \to 0$ as λ goes to 0. Then, we are in a position to pass to the limit as λ tends to 0 in (32), and this yields, for any $\boldsymbol{\varphi} \in \mathbb{F}$,

$$(\boldsymbol{h}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi})_2 - \boldsymbol{F}_0((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}) = \langle \boldsymbol{b}_0, \boldsymbol{\varphi}\rangle_{-1,1}. \tag{34}$$

Let us now establish the uniqueness of the weak limit. Let $\boldsymbol{h}$ and $\boldsymbol{h}'$ be two possible weak limits of two subsequences of $(\boldsymbol{v}_\lambda)_\lambda$, and $\boldsymbol{F}_0$,$\boldsymbol{F}_0'$ the corresponding linear forms defined as described above. Then (34) provides us with he following equality:

$$\forall \boldsymbol{\varphi} \in \mathbb{F}, \quad (\boldsymbol{h} - \boldsymbol{h}', (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}) = [\boldsymbol{F}_0 - \boldsymbol{F}_0']((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}). \tag{35}$$

Using the antisymmetry of the operator D_t again, we obtain

$$\boldsymbol{F}_\lambda((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\mu) = (D_t\boldsymbol{v}_\lambda, \boldsymbol{v}_\mu)_2 = -(D_t\boldsymbol{v}_\mu, \boldsymbol{v}_\lambda)_2 = -\boldsymbol{F}_\mu((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda).$$

Let us first pass to the limit as λ goes to 0 along the first subsequence, and then pass to the limit as μ goes to 0 along the second subsequence, we obtain

$$\boldsymbol{F}_0(\boldsymbol{h}') = -\boldsymbol{F}_0'(\boldsymbol{h}).$$

Now, it just remains to choose $(-\widetilde{\boldsymbol{S}})^{1/2}\varphi = \boldsymbol{h} - \boldsymbol{h}'$ in (35) and this yields

$$|\boldsymbol{h} - \boldsymbol{h}'|_2^2 = -\boldsymbol{F}_0(\boldsymbol{h}') - \boldsymbol{F}_0'(\boldsymbol{h}) = 0.$$

Hence the weak convergence holds for the whole family. Let us now tackle the strong convergence of $(\boldsymbol{v}_\lambda)_\lambda$. Choosing $\boldsymbol{\varphi} = \boldsymbol{v}_\lambda$ in (34), using $\boldsymbol{F}_0(\boldsymbol{h}) = 0$ and passing to the limit a λ goes to 0, this yields

$$(\boldsymbol{h},\boldsymbol{h})_2 = \lim_{\lambda\to 0}\langle \boldsymbol{b}_0, \boldsymbol{v}_\lambda\rangle_{-1,1} = \lim_{\lambda\to 0}\langle \boldsymbol{b}_\lambda, \boldsymbol{v}_\lambda\rangle_{-1,1} = \lim_{\lambda\to 0}\big[\lambda|\boldsymbol{v}_\lambda|_2^2 + \|\boldsymbol{v}_\lambda\|_1^2\big]. \tag{36}$$

In particular, $|\boldsymbol{h}|_2 = \lim_{\lambda\to 0}|(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda|_2$. Thus, the convergence of the norms implies the strong convergence of the sequence $((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{v}_\lambda)_\lambda$ to $\boldsymbol{h}$ in $L^2(\Omega)$. As a bypass, (36) also implies the convergence of $\big(\lambda|\boldsymbol{v}_\lambda|_2^2\big)_\lambda$ to 0. □

Lemma 5.10. *For each function $\boldsymbol{b} \in \mathbb{H}_{-1}$, there exists a family $(\boldsymbol{b}_\lambda)_\lambda$ of functions in $L^2(\Omega) \cap \mathrm{Dom}(D_t) \cap \mathbb{H}_{-1}$ such that $\|\boldsymbol{b} - \boldsymbol{b}_\lambda\|_{-1}$ converges to 0 as λ goes to 0.*

Proof: Let us consider the solution $\boldsymbol{w}_\lambda \in \mathbb{H}$ of the equation $\lambda\boldsymbol{w}_\lambda - \widetilde{\boldsymbol{S}}\boldsymbol{w}_\lambda = \boldsymbol{b}$ (see Proposition 5.5). Then, for any $\varphi \in \mathcal{C}$,

$$\begin{aligned}(\lambda\boldsymbol{w}_\lambda, \boldsymbol{\varphi})_2 &= \int_{\mathbb{R}^+\times\mathbb{R}} \lambda(\lambda + x)^{-1}\, dE_{\boldsymbol{b},\boldsymbol{\varphi}}(dx, dy)\\ &\leq \Big(\int_{\mathbb{R}^+\times\mathbb{R}} \lambda^2 x^{-1}(\lambda + x)^{-2}\, dE_{\boldsymbol{b},\boldsymbol{b}}(dx, dy)\Big)^{1/2}\|\boldsymbol{\varphi}\|_1.\end{aligned}$$

Since $\boldsymbol{b} \in \mathbb{H}_{-1}$, we have $\int_{\mathbb{R}^+\times\mathbb{R}} x^{-1}\, dE_{\boldsymbol{b},\boldsymbol{b}}(dx, dy) < \infty$. Thus the Lebesgue theorem ensures that the above integral converges to 0 as λ goes to 0. Hence, $\|\lambda\boldsymbol{w}_\lambda\|_{-1}$ converges to 0 as λ goes to 0. We can now choose a family $(\boldsymbol{\varphi}_\lambda)_\lambda$ in $\mathcal{C}$ such that $\|\boldsymbol{w}_\lambda - \boldsymbol{\varphi}_\lambda\|_1 \to 0$ as λ goes to 0. Finally,

$$\|\boldsymbol{b} - \widetilde{\boldsymbol{S}}\boldsymbol{\varphi}_\lambda\|_{-1} \leq \|\boldsymbol{b} - \widetilde{\boldsymbol{S}}\boldsymbol{w}_\lambda\|_{-1} + \|\widetilde{\boldsymbol{S}}\boldsymbol{w}_\lambda - \widetilde{\boldsymbol{S}}\boldsymbol{\varphi}_\lambda\|_{-1} \leq \|\lambda\boldsymbol{w}_\lambda\|_{-1} + \|\boldsymbol{w}_\lambda - \boldsymbol{\varphi}_\lambda\|_1$$

also converges to 0 as λ tends to 0 and, clearly, $\widetilde{\boldsymbol{S}}\boldsymbol{\varphi}_\lambda \in L^2(\Omega) \cap \mathrm{Dom}(D_t)$. □

Lemma 5.11. *Let $(\boldsymbol{b}_\lambda)_\lambda$ and $(\boldsymbol{b}'_\lambda)_\lambda$ be two families in $\mathbb{H}_{-1}$ such that $\|\boldsymbol{b}_\lambda - \boldsymbol{b}'_\lambda\|_{-1} \to 0$ as λ goes to 0. Let $(\boldsymbol{v}_\lambda)_\lambda$ and $(\boldsymbol{v}'_\lambda)_\lambda$ two families in $\mathbb{F}$ solving equation* (32) *with respectively $\boldsymbol{b}_\lambda$ and $\boldsymbol{b}'_\lambda$ as right-hand side. Then $\lambda|\boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda|_2^2 + \|\boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda\|_1^2 \to 0$ as λ goes to 0.*

Proof: Making the difference between the two equations corresponding to $\boldsymbol{v}_\lambda$ and $\boldsymbol{v}'_\lambda$, this yields for any $\varphi \in \mathbb{F}$,

$$\lambda(\boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda, \boldsymbol{\varphi})_2 + \langle \boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda, \boldsymbol{\varphi}\rangle_1 - (D_t\boldsymbol{v}_\lambda - D_t\boldsymbol{v}'_\lambda, \boldsymbol{\varphi})_2 = \langle \boldsymbol{b}_\lambda - \boldsymbol{b}'_\lambda, \boldsymbol{\varphi}\rangle_{-1,1}.$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda$, we easily deduce $\lambda|\boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda|_2^2 + \|\boldsymbol{v}_\lambda - \boldsymbol{v}'_\lambda\|_1 \leq \|\boldsymbol{b}_\lambda - \boldsymbol{b}'_\lambda\|_{-1}$. The result follows. □

Let us now investigate the general case, that means that we aim at replacing $\widetilde{\boldsymbol{S}}$ by $\boldsymbol{L}$ in Proposition 5.9. We first set out the main ideas of the proof. Let us formally write

$$\begin{aligned}\lambda - \boldsymbol{L} - D_t &= \lambda - \widetilde{\boldsymbol{S}} - D_t - (\boldsymbol{L} - \widetilde{\boldsymbol{S}})\\ &= \big(\mathrm{I} - \big[\boldsymbol{L} - \widetilde{\boldsymbol{S}}\big](\lambda - \widetilde{\boldsymbol{S}} - D_t)^{-1}\big)(\lambda - \widetilde{\boldsymbol{S}} - D_t)\end{aligned}$$

If we can prove that $\big[\boldsymbol{L}-\widetilde{\boldsymbol{S}}\big](\lambda-\widetilde{\boldsymbol{S}}-D_t)^{-1}$ defines a strictly contractive operator, then we will be in position to inverse it. It turns out that it is actually bounded but not strictly contractive. To overcome this difficulty, we introduce a small parameter δ to make the operator $\delta\big[\boldsymbol{L} - \widetilde{\boldsymbol{S}}\big](\lambda - \widetilde{\boldsymbol{S}} - D_t)^{-1}$ strictly contractive. Then, an iteration procedure proves that δ can be chosen equal to 1.

Proposition 5.12. *Let $(\boldsymbol{b}_\lambda)_{\lambda>0}$ be a family of functions in $\mathbb{H}_{-1}$ that is strongly convergent in $\mathbb{H}_{-1}$ to some $\boldsymbol{b}_0 \in \mathbb{H}_{-1}$ and bounded in $\mathcal{H}$. Then there exists $\delta_0 > 0$ such that, for any $0 \le \delta \le \delta_0$, for any $\lambda > 0$, the solution (in the sense of Proposition 5.4) $\boldsymbol{u}_\lambda \in \mathbb{F}$ (with $D_t\boldsymbol{u}_\lambda \in \mathbb{H}$) of the equation*

$$\lambda \boldsymbol{u}_\lambda - \delta \boldsymbol{L}\boldsymbol{u}_\lambda - (1-\delta)\widetilde{\boldsymbol{S}}\boldsymbol{u}_\lambda - D_t\boldsymbol{u}_\lambda = \boldsymbol{b}_\lambda,$$

satisfies: $\exists \boldsymbol{\eta} \in L^2(\Omega)$ such that $\lambda|\boldsymbol{u}_\lambda|_2^2 + |(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}_\lambda - \boldsymbol{\eta}|_2 \to 0$ as λ goes to 0.

Proof: Consider the operator $P_\lambda : \mathcal{H} \to \mathcal{H}$ defined by $P_\lambda(\boldsymbol{b}) = (\boldsymbol{L} - \widetilde{\boldsymbol{S}})(\lambda - \widetilde{\boldsymbol{S}} - D_t)^{-1}(\boldsymbol{b})$. Note that Proposition 5.4 applies for all coefficients $\boldsymbol{a}$ and $\boldsymbol{H}$ satisfying Assumption 2.3. In particular, it works for $\boldsymbol{a} = \widetilde{\boldsymbol{a}}$ and $\boldsymbol{H} = 0$, so that P_λ is well defined. Lemma 5.13 below proves that $\|P_\lambda\|_{\mathcal{H}\to\mathcal{H}}$ is bounded with a norm that only depends on the constants M, C_1^H, C_2^a and C_2^H (see Assumption 2.3). Therefore, we can choose $\delta_0 > 0$ such that $\|\delta_0 P_\lambda\|_{\mathcal{H}\to\mathcal{H}} < 1$ (actually $\delta_0 < \big[2(2 + M + C_1^H)(1 + C_2^a + C_2^H)\big]^{-1}$). For $0 < \delta < \delta_0$, we can then define the operator $[\mathrm{I} - \delta P_\lambda]^{-1} : \mathcal{H} \longrightarrow \mathcal{H}$. Note that $(\lambda - \delta\boldsymbol{L} - (1-\delta)\widetilde{\boldsymbol{S}} - D_t)^{-1} = (\lambda - \widetilde{\boldsymbol{S}} - D_t)^{-1}\big[\mathrm{I} - \delta P_\lambda\big]^{-1}$. Thanks to Proposition 5.9, it is sufficient to prove that $\big[\mathrm{I} - \delta P_\lambda\big]^{-1}(\boldsymbol{b}_\lambda)$ is convergent in $\mathbb{H}_{-1}$. But $\big[\mathrm{I} - \delta P_\lambda\big]^{-1}(\boldsymbol{b}_\lambda) = \sum_{n=0}^{\infty}(\delta P_\lambda)^n(\boldsymbol{b}_\lambda)$. Lemma 5.13 ensures that the sum converges uniformly with respect to $\lambda > 0$. It just remains to prove that, for each fixed $n \ge 0$, $((\delta P_\lambda)^n(\boldsymbol{b}_\lambda))_\lambda$ converges in $\mathbb{H}_{-1}$. This can be proved by induction on $n \in \mathbb{N}$. For $n = 0$, $(\boldsymbol{b}_\lambda)_{\lambda>0}$ is convergent by assumption. Then, if the family $((\delta P_\lambda)^n(\boldsymbol{b}_\lambda))_\lambda$ is convergent in $\mathbb{H}_{-1}$, we can apply Proposition 5.9 to ensure that the family $\big((-\widetilde{\boldsymbol{S}})^{1/2}(\lambda - \widetilde{\boldsymbol{S}} - D_t)^{-1}[(\delta P_\lambda)^n(\boldsymbol{b}_\lambda)]\big)_\lambda$ converges in $L^2(\Omega)$. This implies the convergence of $((\delta P_\lambda)^{n+1}(\boldsymbol{b}_\lambda))_\lambda$ in $\mathbb{H}_{-1}$. □

Lemma 5.13. *The norms of $P_\lambda : (\mathcal{H}, \|\cdot\|_{-1}) \to (\mathbb{H}_{-1}, \|\cdot\|_{-1})$ and $P_\lambda : (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \to (\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ are both bounded from above by $2(2 + M + C_1^H)(1 + C_2^a + C_2^H)$.*

Proof : Fix $\boldsymbol{b} \in \mathcal{H}$. Let $\boldsymbol{u}_\lambda \in \mathbb{F}$ (with $D_t\boldsymbol{u}_\lambda \in \mathbb{H}$) be the solution of the equation (apply Proposition 5.4 with $\boldsymbol{a} = \widetilde{\boldsymbol{a}}$, $\boldsymbol{H} = 0$, $\boldsymbol{h} = 0$ and $m = 1$)

$$\forall \boldsymbol{\varphi} \in \mathbb{F}, \quad \lambda(\boldsymbol{u}_\lambda, \boldsymbol{\varphi})_2 + \langle \boldsymbol{u}_\lambda, \boldsymbol{\varphi}\rangle_1 - (D_t\boldsymbol{u}_\lambda, \boldsymbol{\varphi})_2 = \langle \boldsymbol{b}, \boldsymbol{\varphi}\rangle_{-1,1}.$$

It derives from (20a) that $\lambda|\boldsymbol{u}_\lambda|_2^2 + \|\boldsymbol{u}_\lambda\|_1^2 \le \|\boldsymbol{b}\|_{-1}^2$, in such a way that

$$\|P_\lambda(\boldsymbol{b})\|_{-1} = \|(\boldsymbol{L} - \widetilde{\boldsymbol{S}})\boldsymbol{u}_\lambda\|_{-1} \le (1 + M + C_1^H)\|\boldsymbol{u}_\lambda\|_1 \le (1 + M + C_1^H)\|\boldsymbol{b}\|_{-1}.$$

This proves the first point.
Consider now $\boldsymbol{u} \in \mathbb{F}$ with $D_t\boldsymbol{u} \in \mathbb{H}$. An easy computation proves that, for any $s \in \mathbb{R}^*$ and $\boldsymbol{\varphi} \in \mathcal{C}$,

$$\begin{aligned}
\boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}, (-\widetilde{\boldsymbol{S}})^{1/2}\Lambda_s\boldsymbol{\varphi}\big) &= -\Lambda_{-s}\boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}, (-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{\varphi}\big)\\
&\quad - \boldsymbol{T}_a\big((-\widetilde{\boldsymbol{S}})^{1/2}\Lambda_s\boldsymbol{u}, (-\widetilde{\boldsymbol{S}})^{1/2}T_{s,0}\boldsymbol{\varphi}\big)\\
&\le C_2^a\|\boldsymbol{u}\|_1\|\boldsymbol{\varphi}\|_1 + M\|D_t\boldsymbol{u}\|_1\|\boldsymbol{\varphi}\|_1.
\end{aligned} \tag{37}$$

In the above inequalities, we use $\|\boldsymbol{u}\|_1 = \|T_{s,0}\boldsymbol{u}\|_1$ and $\|\Lambda_s\boldsymbol{u}\|_1 \le \|D_t\boldsymbol{u}\|_1$. This latter point can be proved for $\boldsymbol{u} \in \mathcal{C}$ as follows

$$\|\Lambda_s\boldsymbol{u}\|_1^2 = -(\Lambda_s\boldsymbol{u}, \widetilde{\boldsymbol{S}}\Lambda_s\boldsymbol{u})_2 = -\int_0^1\int_0^1 (D_tT_{r,0}\boldsymbol{u}, \widetilde{\boldsymbol{S}}D_tT_{u,0}\boldsymbol{u})_2\,dr\,du \le -(D_t\boldsymbol{u}, \widetilde{\boldsymbol{S}}D_t\boldsymbol{u})_2.$$

The general case is treated by density arguments.
As in (37), we have $\boldsymbol{T}_H\big((-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u},(-\widetilde{\boldsymbol{S}})^{1/2}\Lambda_s\boldsymbol{\varphi}\big)\leq C_2^H\|\boldsymbol{u}\|_1\|\boldsymbol{\varphi}\|_1+C_1^H\|D_t\boldsymbol{u}\|_1\|\boldsymbol{\varphi}\|_1$. Hence,

$$\|(\boldsymbol{L}-\widetilde{\boldsymbol{S}})(\boldsymbol{u})\|_T\leq(C_2^H+C_2^a)\|\boldsymbol{u}\|_1+(C_1^H+M+1)\|D_t\boldsymbol{u}\|_1.$$

Then, Proposition 5.4 ensures that $D_t\boldsymbol{u}_\lambda\in\mathbb{H}$ and $\|D_t\boldsymbol{u}_\lambda\|_1\leq 2\|\boldsymbol{b}\|_T+2(C_2^H+C_2^a)\|\boldsymbol{b}\|_{-1}$ (see (20b)) so that we finally obtain

$$\|P_\lambda(\boldsymbol{b})\|_T\leq(C_2^H+C_2^a)\|\boldsymbol{b}\|_{-1}+2(C_1^H+M+1)\big(\|\boldsymbol{b}\|_T+(C_2^H+C_2^a)\|\boldsymbol{b}\|_{-1}\big). \tag{38}$$

The result follows. □

Proof of Proposition 5.7: The last step before proving Proposition 5.7 consists in lifting the restriction of the smallness of δ_0. The previous construction provides us with δ_0 strictly less than 1. We perform an induction to get round this restriction whose initialization is the construction of δ_0. The second step consists in iterating our arguments to the operator

$$\begin{aligned}&\lambda-(\delta_0+\delta_1)\boldsymbol{L}-(1-\delta_0-\delta_1)\widetilde{\boldsymbol{S}}-D_t\\&\qquad=\big[\mathrm{I}-\delta_1(\boldsymbol{L}-\widetilde{\boldsymbol{S}})[\lambda-\delta_0\boldsymbol{L}-(1-\delta_0)\widetilde{\boldsymbol{S}}-D_t]^{-1}\big](\lambda-\delta_0\boldsymbol{L}-(1-\delta_0)\widetilde{\boldsymbol{S}}-D_t).\end{aligned}$$

We exactly repeat the proof of Proposition 5.12 except that the operator $\lambda-(1-\delta_0-\delta_1)\widetilde{\boldsymbol{S}}-(\delta_0+\delta_1)\boldsymbol{L}-D_t$ plays the role of the operator $\lambda-(1-\delta_0)\widetilde{\boldsymbol{S}}-\delta_0\boldsymbol{L}-D_t$ and we apply Proposition 5.12 with the operator $\lambda-(1-\delta_1)\widetilde{\boldsymbol{S}}-\delta_1\boldsymbol{L}-D_t$ instead of applying Proposition 5.9 with $\lambda-\widetilde{\boldsymbol{S}}-D_t$. Of course, a restriction about the smallness of δ_1 is imposed by this procedure. Even if it means substituting $\widetilde{a}$ with $m\widetilde{a}$, we assume, without loss of generality, that $m=1$. Thus Lemma 5.13 remains valid for the operator $P_\lambda^1:\mathcal{H}\to\mathcal{H}$ defined by $P_\lambda^1(\boldsymbol{b})=(\boldsymbol{L}-\widetilde{\boldsymbol{S}})(\lambda-(1-\delta_0)\widetilde{\boldsymbol{S}}-\delta_0\boldsymbol{L}-D_t)^{-1}(\boldsymbol{b})$. This is of the utmost importance because that means that we can choose $\delta_1=\delta_0$. Thus we can iterate these arguments until we find δ_N such that $\delta_0+\delta_1+\cdots+\delta_N>1$ and such that Proposition 5.12 still holds except that $\delta<\delta_0$ is everywhere replaced by $\delta<\delta_0+\delta_1+\cdots+\delta_N$. Proposition 5.7 follows. □

Now let us prove that the drift $\boldsymbol{b}$ of the diffusion process X fulfills the assumptions of Proposition 5.7. To this purpose, let us establish

Lemma 5.14. *For each $i\in\{1,\dots,d\}$, $\boldsymbol{b}_i$ belongs to $\mathbb{H}_{-1}$ and $\forall s\in\mathbb{R},\forall\boldsymbol{\varphi}\in\mathcal{C}$,*

$$\langle\boldsymbol{b}_i,\Lambda_s\boldsymbol{\varphi}\rangle_{-1,1}\leq(C_2^a+C_2^H)|(\widetilde{\boldsymbol{a}}E_i,E_i)_2|^{1/2}\|\boldsymbol{\varphi}\|_1.$$

Proof: Let $(E_1,\dots,E_d)$ be the canonical basis of $\mathbb{R}^d$. Then we have

$$\begin{aligned}(\boldsymbol{b}_i,\boldsymbol{\varphi})_2&=1/2\sum_j\big(e^{2\boldsymbol{V}}D_j(e^{-2\boldsymbol{V}}[\boldsymbol{a}+\boldsymbol{H}]_{ij}),\boldsymbol{\varphi}\big)_2\\&=-1/2\big([\boldsymbol{a}-\boldsymbol{H}]D\boldsymbol{\varphi},E_i\big)_2\\&\leq 1/2\big|\big(\boldsymbol{a}D\boldsymbol{\varphi},E_i\big)_2\big|+1/2\big|\big(\boldsymbol{H}D\boldsymbol{\varphi},E_i\big)_2\big|\\&\overset{Cauchy-Schwarz}{\leq}M\|\boldsymbol{\varphi}\|_1|(\widetilde{\boldsymbol{a}}E_i,E_i)_2|^{1/2}+C_1^H\|\boldsymbol{\varphi}\|_1|(\widetilde{\boldsymbol{a}}E_i,E_i)_2|^{1/2}\end{aligned}$$

and this proves the first point. Then, $\forall s>0,\forall\boldsymbol{\varphi}\in\mathcal{C}$, we have

$$\begin{aligned}\langle\boldsymbol{b}_i,\Lambda_s\boldsymbol{\varphi}\rangle_{-1,1}&=-(1/2)\big([\boldsymbol{a}+\boldsymbol{H}]E_i,\Lambda_sD\boldsymbol{\varphi}\big)_2\\&=(1/2)\big(\Lambda_{-s}[\boldsymbol{a}+\boldsymbol{H}]E_i,D\boldsymbol{\varphi}\big)_2\\&\overset{\text{Assumption 2.3}}{\leq}(C_2^a+C_2^H)|(\widetilde{\boldsymbol{a}}E_i,E_i)_2|^{1/2}\|\boldsymbol{\varphi}\|_1\quad\square\end{aligned}$$

6 Itô's formula

We are not in a lucky situation of working on an explicit Dirichlet form connected with the generator in $L^2(\Omega,\pi)$ of Y, wrongly denoted by $[\boldsymbol{L}+D_t]$. This raises the following issue: given a function $\boldsymbol{f}\in L^2(\Omega)$ and the function $\boldsymbol{u}_\lambda$ that weakly solves (see Proposition 5.4) $\lambda\boldsymbol{u}_\lambda-(\boldsymbol{L}+D_t)\boldsymbol{u}_\lambda=\boldsymbol{f}$, does the "Ito formula" apply to $\boldsymbol{u}_\lambda$ and to the process Y. Indeed, it is not clear that the construction of $\boldsymbol{u}_\lambda$ in Proposition 5.4 belongs to the domain of the generator of Y. The key tool is the regular approximation $(\boldsymbol{u}_{\lambda,\delta})_\delta$ provided by Proposition 5.4 for a suitable function $\boldsymbol{f}$.
Let us consider a standard 1-dimensional Brownian motion $\{B'_t;t\geq 0\}$ that is independent of $\{B_t;t\geq 0\}$ in such a way that $\{(B'_t,B_t);t\geq 0\}$ is a standard $d+1$-dimensional Brownian motion. Define then the $d+1$-dimensional diffusion process $X^{\omega,\delta}$, starting from 0, as the solution of the SDE:

$$X^{\omega,\delta}_t=\int_0^t\begin{bmatrix}1\\ b(X^{\omega,\delta}_r,\omega)\end{bmatrix}dr+\int_0^t\begin{bmatrix}\sqrt{\delta} & 0\\ 0 & \sigma(X^{\omega,\delta}_r,\omega)\end{bmatrix}d(B',B)_r. \tag{39}$$

The associated diffusion in random medium Y^δ defined by $Y^\delta_t(\omega)=\tau_{X^{\omega,\delta}_t}\omega$ is a Ω-valued Markov process, which admits π as invariant measure (similar to section 4). It also defines a continuous semi-group on $L^2(\Omega)$. The associated (non-symmetric) Dirichlet form is given by (19) (with $\theta=1$) with domain $\mathbb{F}\times\mathbb{F}$ and satisfies a weak sector condition (see [12, Ch. 1, Sect 2.] for the definition). The generator $\boldsymbol{L}^\delta$ is defined on $\mathrm{Dom}(\boldsymbol{L}^\delta)=\{\boldsymbol{u}\in\mathbb{F};B_{\lambda,\delta}(\boldsymbol{u},\cdot)$ is $L^2(\Omega)$-continuous$\}$ (see [12, Ch. 1, Sect 2.] for further details). It coincides on $\mathcal{C}$ with $\boldsymbol{L}+D_t+(\delta/2)D_t^2$. Since b and σ are globally Lipschitz (Assumption 2.2), classical tools of SDE theory ensures that

$$\int_\Omega\mathbb{E}\big[\sup_{0\leq t\leq T}|(t,X^\omega_t)-X^{\omega,\delta}_t|^2\big]\,d\pi\to 0\text{ as }\delta\text{ goes to }0, \tag{40}$$

where both diffusions start from 0.

Proposition 6.1. *Let $\boldsymbol{f}\in L^2(\Omega)$ and a family $(\boldsymbol{u}_\lambda)_{\lambda>0}$ in $\mathbb{F}$ such that:*
1) $\forall\boldsymbol{\varphi}\in\mathbb{F}$, $B_\lambda(\boldsymbol{u}_\lambda,\boldsymbol{\varphi})=(\boldsymbol{f},\boldsymbol{\varphi})_2$,
2) for each $\lambda>0$, there exists a sequence $(\boldsymbol{u}_{\lambda,\delta})_{\delta>0}$ in $\mathbb{F}$ that converges in $\mathbb{H}$ towards $\boldsymbol{u}_\lambda$. Moreover $(\boldsymbol{u}_{\lambda,\delta})_{\delta>0}\in\mathrm{Dom}(\boldsymbol{L}^\delta)$ and satisfies $\lambda\boldsymbol{u}_{\lambda,\delta}-\boldsymbol{L}^\delta\boldsymbol{u}_{\lambda,\delta}=\boldsymbol{f}$.
3) for each fixed $\lambda>0$, $(D_t\boldsymbol{u}_{\lambda,\delta})_\delta$ is bounded in $L^2(\Omega)$.
4) each function $\boldsymbol{u}_{\lambda,\delta}$ has continuous trajectories, that is, for μ almost every $\omega\in\Omega$, the function $(t,x)\in\mathbb{R}^{d+1}\mapsto\boldsymbol{u}_{\lambda,\delta}(\tau_{t,x}\omega)$ is continuous.
Then, $\mathbb{P}_\pi$ a.s., the following formula holds

$$\boldsymbol{u}_\lambda(Y_t)=\boldsymbol{u}_\lambda(Y_0)+\int_0^t(\lambda\boldsymbol{u}_\lambda-\boldsymbol{f})(Y_r)\,dr+\int_0^t\nabla^\sigma\boldsymbol{u}^*_\lambda(Y_r)\,dB_r$$

where $\mathbb{P}_\pi$ is the law of the process Y starting with initial distribution π on Ω.

Proof: Since $\boldsymbol{u}_{\lambda,\delta}\in\mathrm{Dom}(\boldsymbol{L}^\delta)$ and $\lambda\boldsymbol{u}_{\lambda,\delta}-\boldsymbol{L}^\delta\boldsymbol{u}_{\lambda,\delta}=\boldsymbol{f}$, we can write (see Lemma 6.2 below)

$$\begin{aligned}&\boldsymbol{u}_{\lambda,\delta}(Y^\delta_t)-\boldsymbol{u}_{\lambda,\delta}(Y^\delta_0)\\ &\qquad=\int_0^t\boldsymbol{L}^\delta\boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dr+\delta^{1/2}\int_0^tD_t\boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dB'_r+\int_0^t\nabla^\sigma\boldsymbol{u}^*_{\lambda,\delta}(Y^\delta_r)\,dB_r\\ &\qquad=\int_0^t[\lambda\boldsymbol{u}_{\lambda,\delta}-\boldsymbol{f}](Y^\delta_r)\,dr+\delta^{1/2}\int_0^tD_t\boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dB'_r+\int_0^t\nabla^\sigma\boldsymbol{u}^*_{\lambda,\delta}(Y^\delta_r)\,dB_r.\end{aligned} \tag{41}$$

Thanks to (40), the convergence, as $\delta \to 0$, of $(\boldsymbol{u}_{\lambda,\delta})_{\lambda,\delta}$ towards $\boldsymbol{u}_\lambda$ in $\mathbb{H}$ and the boundedness of $(D_t\boldsymbol{u}_{\lambda,\delta})_\delta$ in $L^2(\Omega)$, we can pass to the limit in (41) and complete the proof. $\square$

Lemma 6.2. *Keeping the notations of Proposition 6.1, the following formula holds,* $\mathbb{P}_\pi$ *a.s.,*

$$\boldsymbol{u}_{\lambda,\delta}(Y^\delta_t) - \boldsymbol{u}_{\lambda,\delta}(Y^\delta_0) = \int_0^t \boldsymbol{L}^\delta \boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dr + \delta^{1/2}\int_0^t D_t\boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dB'_r + \int_0^t \nabla^\sigma \boldsymbol{u}^*_{\lambda,\delta}(Y^\delta_r)\,dB_r.$$

Proof: Since $\boldsymbol{u}_{\lambda,\delta} \in \mathrm{Dom}(\boldsymbol{L}^\delta)$, the difference $\boldsymbol{u}_{\lambda,\delta}(Y^\delta_t) - \boldsymbol{u}_{\lambda,\delta}(Y^\delta_0) - \int_0^t \boldsymbol{L}^\delta \boldsymbol{u}_{\lambda,\delta}(Y^\delta_r)\,dr$ is a square-integrable continuous $\mathbb{P}_\pi$-martingale, denoted by M^δ_t. Moreover, for a function $\boldsymbol{\varphi} \in \mathcal{C}$, the classical Ito formula yields $\boldsymbol{\varphi}(Y^\delta_t) - \boldsymbol{\varphi}(Y^\delta_0) = \int_0^t \boldsymbol{L}^\delta\boldsymbol{\varphi}(Y^\delta_r)\,dr + \delta^{1/2}\int_0^t D_t\boldsymbol{\varphi}(Y^\delta_r)\,dB'_r + \int_0^t \nabla^\sigma\boldsymbol{\varphi}^*(Y^\delta_r)\,dB_r$. Then the process $t \mapsto \boldsymbol{u}_{\lambda,\delta}(Y^\delta_t) - \boldsymbol{\varphi}(Y^\delta_t)$ is a continuous semimartingale and Theorem 32 in [18, Ch. 2, Sect. 7] (applied with the function $x \in \mathbb{R} \mapsto x^2$) yields $\mathbb{P}_\pi$ a.s.,

$$\begin{aligned}(\boldsymbol{u}_{\lambda,\delta}(Y^\delta_t)-&\boldsymbol{\varphi}(Y^\delta_t))^2\\ &= (\boldsymbol{u}_{\lambda,\delta}(Y^\delta_t) - \boldsymbol{\varphi}(Y^\delta_0))^2 + 2\int_0^t (\boldsymbol{u}_{\lambda,\delta} - \boldsymbol{\varphi})\boldsymbol{L}^\delta(\boldsymbol{u}_{\lambda,\delta} - \boldsymbol{\varphi})(Y^\delta_r)\,dr\\ &+ 2\int_0^t (\boldsymbol{u}_{\lambda,\delta} - \boldsymbol{\varphi})(Y^\delta_r)\big(dM^\delta_r - \delta^{1/2}D_t\boldsymbol{\varphi}(Y^\delta_r)\,dB'_r - \nabla^\sigma\boldsymbol{\varphi}^*(Y^\delta_r)\,dB_r\big)\\ &+ 2\big[M - \int_0^\cdot \delta^{1/2}D_t\boldsymbol{\varphi}(Y^\delta_r)\,dB'_r - \int_0^\cdot \nabla^\sigma\boldsymbol{\varphi}^*(Y^\delta_r)\,dB_r\big]_t,\end{aligned}\tag{42}$$

where $[X]$ stands for the quadratic variations of the martingale X. Integrating with respect to the measure π, the martingale term vanishes and we deduce

$$\mathbb{E}_\pi\big(2\big[M - \int_0^\cdot \delta^{1/2}D_t\boldsymbol{\varphi}(Y^\delta_r)\,dB'_r - \int_0^\cdot \nabla^\sigma\boldsymbol{\varphi}^*(Y^\delta_r)\,dB_r\big]_t\big) \le 2B_{\lambda,\delta}(\boldsymbol{u}_{\lambda,\delta} - \boldsymbol{\varphi}, \boldsymbol{u}_{\lambda,\delta} - \boldsymbol{\varphi}).\tag{43}$$

Choosing a sequence $(\boldsymbol{\varphi}_n)_n$ in $\mathcal{C}$ that converges in $\mathbb{F}$ towards $\boldsymbol{u}_{\lambda,\delta}$, we easily complete the proof with the help of (43). $\square$

Note that the time reversed process $t \mapsto Y^\delta_{T-t}$ is a Markov process with respect to the backward filtration $(\mathcal{G}^\delta_t)_{0\le t\le T}$, where $\mathcal{G}^\delta_s$ is the σ-algebra on Ω generated by $\big\{Y^\delta_r; t \le r \le T\big\}$, and admits the adjoint operator $(\boldsymbol{L}^\delta)^*$ of $\boldsymbol{L}^\delta$ in $L^2(\Omega,\pi)$ as generator, which coincides on $\mathcal{C}$ with $\boldsymbol{L}^* - D_t + (\delta/2)D^2_t$. From (40), $t \mapsto Y^\delta_{T-t}$ approximates the process $t \mapsto Y_{T-t}$ as δ tends to 0. It is then readily seen that we can slightly modify the proof of Proposition 6.1 and prove the

Proposition 6.3. *Let* $\boldsymbol{f} \in L^2(\Omega)$ *and a family* $(\boldsymbol{u}_\lambda)_{\lambda>0}$ *in* $\mathbb{F}$ *such that:*
1) $\forall\boldsymbol{\varphi} \in \mathbb{F}$, $B_\lambda(\boldsymbol{\varphi}, \boldsymbol{u}_\lambda) = (\boldsymbol{f}, \boldsymbol{\varphi})_2$,
2) for each $\lambda > 0$, *there exists a sequence* $(\boldsymbol{u}_{\lambda,\delta})_{\delta>0}$ *in* $\mathbb{F}$ *that converges in* $\mathbb{H}$ *towards* $\boldsymbol{u}_\lambda$. *Moreover* $(\boldsymbol{u}_{\lambda,\delta})_{\delta>0} \in \mathrm{Dom}(\boldsymbol{L}^\delta)^*$ *and satisfies* $\lambda\boldsymbol{u}_{\lambda,\delta} - (\boldsymbol{L}^\delta)^*\boldsymbol{u}_{\lambda,\delta} = \boldsymbol{f}$.
3) for each fixed $\lambda > 0$, $(D_t\boldsymbol{u}_{\lambda,\delta})_\delta$ *is bounded in* $L^2(\Omega)$.
4) each function $\boldsymbol{u}_{\lambda,\delta}$ *has continuous trajectories, that is, for* μ *almost every* $\omega \in \Omega$, *the function* $(t, x) \in \mathbb{R}^{d+1} \mapsto \boldsymbol{u}_{\lambda,\delta}(\tau_{t,x}\omega)$ *is continuous.*
Then, $\mathbb{P}_\pi$ *a.s., the following formula holds*

$$\boldsymbol{u}_\lambda(Y_T - t) = \boldsymbol{u}_\lambda(Y_T) + \int_0^t (\lambda\boldsymbol{u}_\lambda - \boldsymbol{f})(Y_{T-r})\,dr + (M_t - M_0)$$

where M *is a martingale with respect to the backward filtration* $(\mathcal{G}_t)_{0\le t\le T}$, *and* $\mathcal{G}_s$ *is the* σ*-algebra on* Ω *generated by* $\{Y_r; t \le r \le T\}$. *Moreover, the quadratic variations of* M *exactly match* $\int_0^t \nabla^\sigma\boldsymbol{u}^*_\lambda \cdot \nabla^\sigma\boldsymbol{u}_\lambda(Y_{T-r})\,dr$.

7 Ergodic Theorem

Let us now exploit the ergodic properties of the operator $\widetilde{\boldsymbol{S}}$ stated in Assumption 2.4 and prove

Theorem 7.1. *Let $\boldsymbol{f} \in L^1(\Omega)$. Then*

$$\mathbb{E}_\pi\Big|\frac{1}{t}\int_0^t \boldsymbol{f}(Y_r)\,dr - \pi(\boldsymbol{f})\Big| \to 0 \text{ as } t \text{ goes to } \infty.$$

Proof: We suppose at first that $\boldsymbol{f} \in \mathcal{C}$. Even if it means considering $\boldsymbol{f} - \pi(\boldsymbol{f})$ instead of $\boldsymbol{f}$, we assume that $\pi(\boldsymbol{f}) = 0$. Clearly, $\boldsymbol{f} \in \mathrm{Dom}(D_t)$ and Proposition 5.4 applies. For each $\lambda > 0$, it provides us with a function $\boldsymbol{u}_\lambda \in \mathbb{F}$ such that

$$\forall \boldsymbol{\varphi} \in \mathbb{F}, \quad B_\lambda(\boldsymbol{u}_\lambda, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi})_2. \tag{44}$$

Moreover, (20a) and (20b) ensures that the families $(\lambda \boldsymbol{u}_\lambda)_\lambda$, $(\lambda D_t \boldsymbol{u}_\lambda)_\lambda$ and $(\lambda^{1/2}(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}_\lambda)_\lambda$ are bounded in $L^2(\Omega)$. Even if it means considering a subsequence, we assume that $(\lambda \boldsymbol{u}_\lambda)_\lambda$, $(\lambda D_t \boldsymbol{u}_\lambda)_\lambda$ and $(\lambda^{1/2}(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{u}_\lambda)_\lambda$ weakly converge respectively to $\boldsymbol{g}$, $\boldsymbol{g}'$ and $\boldsymbol{G}$ in $L^2(\Omega)$. Since the operator D_t is closed, it turns out that $\boldsymbol{g}' = D_t \boldsymbol{g}$. Let us now prove now that $\boldsymbol{g} \in \mathrm{Dom}(\boldsymbol{L})$. Consider $\boldsymbol{\varphi} \in \mathrm{Dom}(\boldsymbol{L}^*)$. Then we derive from (44) that

$$\lambda(\boldsymbol{f}, \boldsymbol{\varphi})_2 = \lambda B_\lambda(\boldsymbol{u}_\lambda, \boldsymbol{\varphi}) = \lambda^2(\boldsymbol{u}_\lambda, \boldsymbol{\varphi})_2 - (\lambda \boldsymbol{u}_\lambda, \boldsymbol{L}^*\boldsymbol{\varphi})_2 - (\lambda D_t \boldsymbol{u}_\lambda, \boldsymbol{\varphi})_2.$$

Passing to the limit as λ goes to 0, we deduce $(\boldsymbol{g}, \boldsymbol{L}^*\boldsymbol{\varphi})_2 = -(D_t\boldsymbol{g}, \boldsymbol{\varphi})_2$. Hence $\boldsymbol{g} \in \mathrm{Dom}(\boldsymbol{L}^{**}) = \mathrm{Dom}(\boldsymbol{L}) \subset \mathbb{H}$ and $\boldsymbol{L}\boldsymbol{g} = -D_t\boldsymbol{g}$. In particular

$$m\|\boldsymbol{g}\|_1^2 \leq -(\boldsymbol{g}, \boldsymbol{L}\boldsymbol{g})_2 = (D_t\boldsymbol{g}, \boldsymbol{g})_2 = 0$$

so that $\boldsymbol{g} \in \mathrm{Dom}((-\widetilde{\boldsymbol{S}})^{1/2})$ and $(-\widetilde{\boldsymbol{S}})^{1/2}\boldsymbol{g} = 0$. As a consequence, $\boldsymbol{g} \in \mathrm{Dom}(\widetilde{\boldsymbol{S}})$ and $\widetilde{\boldsymbol{S}}\boldsymbol{g} = 0$. From Assumption 2.4, $\boldsymbol{g}$ is invariant under space translations in such a way that $D_t\boldsymbol{g} = -\boldsymbol{L}\boldsymbol{g} = 0$ and $\boldsymbol{g}$ is also invariant under time translations. Thus the ergodicity of the measure μ implies that $\boldsymbol{g}$ is constant (μ a.s.). Choosing $\boldsymbol{\varphi}$ equal to the constant function $\mathbf{1}$ in (44), we deduce $\boldsymbol{g} = 0$. We now aim at proving that the convergence of $(\lambda \boldsymbol{u}_\lambda)_\lambda$ towards 0 holds in the strong sense. In what follows, we make no distinction between $0 \in \mathbb{R}$ and the constant function that matches 0 over Ω. We just have to write

$$0 = (0, \boldsymbol{f})_2 = \lim_{\lambda\to 0}(\lambda \boldsymbol{u}_\lambda, \boldsymbol{f})_2 = \lim_{\lambda\to 0} B_\lambda(\lambda \boldsymbol{u}_\lambda, \lambda \boldsymbol{u}_\lambda)_2 \geq \limsup_{\lambda\to 0} |\lambda \boldsymbol{u}_\lambda|_2^2.$$

Note now that the approximating family $(\boldsymbol{u}_{\lambda,\delta})_\delta$ provided by Proposition 5.4 is given by $\boldsymbol{u}_{\lambda,\delta}(\omega) = \int_0^\infty e^{-\lambda r}\mathbb{E}_0[f(X_r^{\omega,\delta}, \omega)]\,dr$. For each $(t, x) \in \mathbb{R}^{d+1}$, the law of the process $(t, x) + X^{\tau_{t,x}\omega,\delta}$, $X^{\tau_{t,x}\omega,\delta}$ starting from $0 \in \mathbb{R}^{d+1}$, is the same as the law of the process $X^{\omega,\delta}$ starting from $(t, x) \in \mathbb{R}^{d+1}$ (see the proof at the end of Section 8). Hence $\boldsymbol{u}_{\lambda,\delta}(\tau_{t,x}\omega) = \int_0^\infty e^{-\lambda r}\mathbb{E}_{t,x}[f(X_r^{\omega,\delta}, \omega)]\,dr$. Since $\boldsymbol{f}$ is smooth and $X^{\omega,\delta}$ is a Feller process, $\boldsymbol{u}_{\lambda,\delta}$ has continuous trajectories. Thus Proposition 6.1 applies and it yields

$$\int_0^t \boldsymbol{f}(Y_r)\,dr = (\boldsymbol{u}_\lambda(Y_0^\delta) - \boldsymbol{u}_\lambda(Y_t)) + \int_0^t \lambda \boldsymbol{u}_\lambda(Y_r)\,dr + \int_0^t \nabla^\sigma \boldsymbol{u}_\lambda^*(Y_r)\,dB_r.$$

Thanks to (20a) and the invariance of the measure π for the process Y, we can find a constant C, which depends neither on λ nor on t, such that

$$\mathbb{E}_\pi\Big|\frac{1}{t}\int_0^t \boldsymbol{f}(Y_r)\,dr\Big|^2 \leq C/(t\lambda)^2 + C|\lambda \boldsymbol{u}_\lambda|_2^2 + C/(t\lambda^{1/2}).$$

It just remains to choose λ small enough and then t large enough to complete the proof in the case $\boldsymbol{f} \in \mathcal{C}$. The general case is treated with the density of $\mathcal{C}$ in $L^1(\Omega)$ and the invariance of the measure π. Since it raises no particular difficulty, details are left to the reader. $\square$

8 Invariance principle

Notation :
Up to the end of this paper, for $i \in \{1, \dots, d\}$ we denote by $\boldsymbol{u}^i_\lambda$ the solution of the equation (in the weak sense of Proposition 5.4)

$$\lambda \boldsymbol{u}^i_\lambda - \boldsymbol{L}\boldsymbol{u}^i_\lambda - D_t \boldsymbol{u}^i_\lambda = \boldsymbol{b}_i.$$

From Proposition 5.7, there exists $\boldsymbol{\xi}_i \in (L(\Omega))^d$ such that $\lambda|\boldsymbol{u}^i_\lambda|_2^2 + |\boldsymbol{\xi}_i - \nabla^\sigma \boldsymbol{u}^i_\lambda|_2 \to 0$ as λ goes to 0. $\square$
Applying the Ito formula (see Proposition 6.1) to the function $\boldsymbol{u}_{\varepsilon^2}$ yields

$$\varepsilon X^\omega_{t/\varepsilon^2} = H^{\varepsilon,\omega}_t + \varepsilon \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u^*_\lambda)(r, X^\omega_r, \omega)\, dB_r,$$

where

$$H^{\varepsilon,\omega}_t = \varepsilon^3 \int_0^{t/\varepsilon^2} u_{\varepsilon^2}(r, X^\omega_r, \omega)\, dr - \varepsilon u_{\varepsilon^2}(t/\varepsilon^2, X^\omega_{t/\varepsilon^2}, \omega) + \varepsilon u_{\varepsilon^2}(0,0,\omega).$$

For the reader's convenience, it is worth recalling that $Y_t = \tau_{t, X^\omega_t}$ and $\mathbb{P}_\pi$ is the law of the process Y with initial distribution π. We want to show that the finite dimensional distributions of the process $H^{\varepsilon,\omega}$ converges in $\mathbb{P}_\pi$-probability to 0. Using the Cauchy-Scharz inequality and the invariance of the measure π, we get the estimate

$$\mathbb{E}_\pi[(H^{\varepsilon,\omega}_t)^2] \le 3(2+t^2)\varepsilon^2 |u_{\varepsilon^2}|_2^2$$

and this latter quantity converges to 0 as ε goes to 0.
Let us now investigate the convergence of the process $t \mapsto \varepsilon \int_0^{t/\varepsilon^2} (\sigma + \nabla^\sigma u^*_\lambda)(Y_r)\, dB_r$ whose quadratic variations are given by

$$\begin{aligned}\varepsilon^2 \int_0^{t/\varepsilon^2} (\boldsymbol{\sigma} + \nabla^\sigma \boldsymbol{u}^*_{\varepsilon^2})(\boldsymbol{\sigma} + \nabla^\sigma \boldsymbol{u}^*_{\varepsilon^2})^*(Y_r)\, dr &= \varepsilon^2 \int_0^{t/\varepsilon^2} (\boldsymbol{\sigma} + \boldsymbol{\xi}^*)(\boldsymbol{\sigma} + \boldsymbol{\xi}^*)^*(Y_r)\, dr \\ &+ \Big(\varepsilon^2 \int_0^{t/\varepsilon^2} (\boldsymbol{\sigma} + \nabla^\sigma \boldsymbol{u}^*_{\varepsilon^2})(\boldsymbol{\sigma} + \nabla^\sigma \boldsymbol{u}^*_{\varepsilon^2})^*(Y_r)\, dr - \varepsilon^2 \int_0^{t/\varepsilon^2} (\boldsymbol{\sigma} + \boldsymbol{\xi}^*)(\boldsymbol{\sigma} + \boldsymbol{\xi}^*)^*(Y_r)\, dr\Big).\end{aligned}$$

With the help of Theorem 7.1, the finite dimensional distributions of the former term in the right-hand side converge in $L^1(\mathbb{P}_\pi)$ to the ones of the process $t \mapsto At$, where the matrix A is given by

$$A = \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\xi}^*)(\boldsymbol{\sigma} + \boldsymbol{\xi}^*)^*\, d\pi. \tag{45}$$

The finite dimensional distributions of the latter term in the right-hand side converge in $L^1(\mathbb{P}_\pi)$ to 0. Indeed, after integrating with respect to the probability measure $\mathbb{P}_\pi$, it is bounded by $Ct|\nabla^\sigma\, \boldsymbol{u}_{\varepsilon^2} - \boldsymbol{\xi}|_2^2$. Hence we conclude by applying the central limit theorem for martingales that the finite dimensional distributions of the process $\varepsilon X^\omega_{t/\varepsilon^2}$ converge in law to the ones of the process $A^{1/2}B_t$.

Proposition 8.1. *The process $\varepsilon X^{\omega}_{t/\varepsilon^2}$ is tight in the space $C([0,T];\mathbb{R}^d)$. Hence it converges in law in the space $C([0,T];\mathbb{R}^d)$ towards the process $A^{1/2}B_t$.*

Proof : The next section is devoted to the proof of the tightness $\square$

Let us now to determine the limit when the starting point is not 0 but x/ε.

$$\mathbb{E}_{x/\varepsilon}\big[f(\varepsilon X^{\omega}_{t/\varepsilon^2})\big] = \mathbb{E}_0\big[f(x+\varepsilon X^{\tau(0,x/\varepsilon)\omega}_{t/\varepsilon^2})\big] \overset{\text{in law w.r.t. } \mu}{=} \mathbb{E}_0\big[f(x+\varepsilon X^{\omega}_{t/\varepsilon^2})\big]$$
$$\xrightarrow[\varepsilon\to 0]{\pi \text{ prob}} \mathbb{E}\big[f(x+A^{1/2}B_t)\big]$$

For the first above equality we used the following fact. If

$$X_t = x + \int_0^t b\left(r, X_r, \omega\right)\,dr + \int_0^t \sigma\left(r, X_r, \omega\right)\,dB_r$$

and $Z_t \triangleq X_t - x$ then Z_t solves the SDE

$$Z_t = \int_0^t b\left(r, Z_r, \tau_{(0,x)}\omega\right)\,dr + \int_0^t \sigma\left(r, Z_r, \tau_{(0,x)}\omega\right)\,dB_r,$$

so that the law of the process X^{ω} starting from $x \in \mathbb{R}^d$ is equal to the law of the process $x + X^{\tau_x\omega}$ where $X^{\tau_x\omega}$ is starting from 0. We sum up:

Theorem 8.2. *Let f be a continuous, bounded function on $\mathbb{R}^d$. Then the solution $z(x,t,\omega)$ of the partial differential equation* (2) *with initial condition $z(0,x,\omega) = f(x)$ satisfies the following convergence: $z(x/\varepsilon, t/\varepsilon^2, \omega)$ converges in π-probability as $\varepsilon \to 0$ to $\mathbb{E}\left[f(x+A^{1/2}B_t)\right]$, which is the viscosity solution of the deterministic equation* (3) *with the same initial condition. The matrix A is given by*

$$A = \int_{\Omega} (\boldsymbol{\sigma} + \boldsymbol{\xi}^*)(\boldsymbol{\sigma} + \boldsymbol{\xi}^*)^*\,d\pi.$$

9 Tightness

Let us now investigate the tightness in $C([0,T];\mathbb{R}^d)$ of the process

$$\varepsilon X^{\omega}_{t/\varepsilon^2} = \varepsilon \int_0^{t/\varepsilon^2} b(r, X^{\omega}_r, \omega)\,dr + \varepsilon \int_0^{t/\varepsilon^2} \sigma(r, X^{\omega}_r, \omega)\,dB_r.$$

The tightness of the first term in the above right-hand side is readily derived from the Burkholder-Davis-Gundy inequality and the boundedness of the diffusion coefficient $\boldsymbol{\sigma}$. Concerning the second term, we are going to exploit ideas of [20] or [22].

For any $i \in \{1,\dots,d\}$ and $\lambda > 0$, we put $\boldsymbol{w}_\lambda = (\lambda - \boldsymbol{S})^{-1}\boldsymbol{b}_i \in \mathbb{H} \cap \mathrm{Dom}(\boldsymbol{S})$ (see Proposition 5.5). Proposition 5.4 (with $\theta = 0$ and $\boldsymbol{H} = 0$) also ensures that $\boldsymbol{w}_\lambda \in \mathbb{F}$, $D_t\boldsymbol{w}_\lambda \in \mathbb{H}$. For each fixed $\lambda > 0$, we can find a sequence $(\boldsymbol{\psi}^n_\lambda)_n$ in $\mathcal{C}$ such that $\|\boldsymbol{\psi}^n_\lambda - \boldsymbol{w}_\lambda\|_1 + \|D_t\boldsymbol{\psi}^n_\lambda - D_t\boldsymbol{w}_\lambda\|_1$ converges to 0 as n goes to ∞. Define $\boldsymbol{A}^n_\lambda = (1/2)\sum_{k,l} D_l\big(\boldsymbol{H}_{kl}D_k\boldsymbol{\psi}^n_\lambda\big)$. From Proposition 5.4, we can find two sequences $(\overline{\boldsymbol{v}}^n_\lambda)_n \subset \mathbb{F} \cap \mathrm{Dom}(\boldsymbol{L})$ and $(\underline{\boldsymbol{v}}^n_\lambda)_n \subset \mathbb{F} \cap \mathrm{Dom}(\boldsymbol{L}^*)$ that respectively solve the equations $(\lambda - \boldsymbol{L})\overline{\boldsymbol{v}}^n_\lambda = \boldsymbol{b}_i - \boldsymbol{A}^n_\lambda$ and $(\lambda - \boldsymbol{L}^*)\underline{\boldsymbol{v}}^n_\lambda = \boldsymbol{b}_i + \boldsymbol{A}^n_\lambda$. Moreover, the functions $\overline{\boldsymbol{v}}^n_\lambda$ and $\underline{\boldsymbol{v}}^n_\lambda$ possess a corresponding approximation sequence $(\overline{\boldsymbol{v}}^n_{\lambda,\delta})_{\delta>0}$ and $(\underline{\boldsymbol{v}}^n_{\lambda,\delta})_{\delta>0}$ (see Proposition 5.4), which both have continuous trajectories since $\boldsymbol{b}_i \pm \boldsymbol{A}^n_\lambda$ have. We are then in position to apply Proposition 6.1. For any $0 \le t \le T$ and $\lambda > 0$

$$\begin{aligned}\overline{\boldsymbol{v}}^n_\lambda(Y_t) - \overline{\boldsymbol{v}}^n_\lambda(Y_0) &= \int_0^t [\boldsymbol{L}\overline{\boldsymbol{v}}^n_\lambda + D_t\overline{\boldsymbol{v}}^n_\lambda](Y_r)\,dr + \overline{\mathcal{M}}^{n,\lambda}_t - \overline{\mathcal{M}}^{n,\lambda}_0 \\ &= \int_0^t [\lambda\overline{\boldsymbol{v}}^n_\lambda - \boldsymbol{b}_i + \boldsymbol{A}^n_\lambda + D_t\overline{\boldsymbol{v}}^n_\lambda](Y_r)\,dr + \overline{\mathcal{M}}^{n,\lambda}_t - \overline{\mathcal{M}}^{n,\lambda}_0,\end{aligned}$$

where $\overline{\mathcal{M}}^{n,\lambda}$ is a martingale with respect to the forward filtration $(\mathcal{F}_t)_{0\le t\le T}$, and $\mathcal{F}_t$ is the σ-algebra on Ω generated by $\{Y_r; 0 \le r \le t\}$. From Proposition 6.3, we also have

$$\begin{aligned}\underline{\boldsymbol{v}}^n_\lambda(Y_0) - \underline{\boldsymbol{v}}^n_\lambda(Y_t) &= \int_0^t [\boldsymbol{L}^*\underline{\boldsymbol{v}}^n_\lambda - D_t\underline{\boldsymbol{v}}^n_\lambda](Y_r)\,dr + \underline{\mathcal{M}}^{n,\lambda}_0 - \underline{\mathcal{M}}^{n,\lambda}_t \\ &= \int_0^t [\lambda\underline{\boldsymbol{v}}^n_\lambda - \boldsymbol{b}_i - \boldsymbol{A}^n_\lambda - D_t\underline{\boldsymbol{v}}^n_\lambda](Y_r)\,dr + \underline{\mathcal{M}}^{n,\lambda}_0 - \underline{\mathcal{M}}^{n,\lambda}_t,\end{aligned}$$

where $\underline{\mathcal{M}}^{n,\lambda}$ is a martingale with respect to the backward filtration $(\mathcal{G}_t)_{0\le t\le T}$, and $\mathcal{G}_s$ is the σ-algebra on Ω generated by $\{Y_r; t \le r \le T\}$. Adding up these equalities, we obtain, for any $0 \le t \le T$,

$$\begin{aligned}2\int_0^t \boldsymbol{b}_i(Y_r)\,dr =& [\underline{\boldsymbol{v}}^n_\lambda - \overline{\boldsymbol{v}}^n_\lambda](Y_t) + [\overline{\boldsymbol{v}}^n_\lambda - \underline{\boldsymbol{v}}^n_\lambda](Y_0) + \int_0^t [\lambda(\overline{\boldsymbol{v}}^n_\lambda + \underline{\boldsymbol{v}}^n_\lambda) + D_t(\overline{\boldsymbol{v}}^n_\lambda - \underline{\boldsymbol{v}}^n_\lambda)](Y_r)\,dr \\ &+ \overline{\mathcal{M}}^{n,\lambda}_t - \overline{\mathcal{M}}^{n,\lambda}_0 + \underline{\mathcal{M}}^{n,\lambda}_0 - \underline{\mathcal{M}}^{n,\lambda}_t\end{aligned}$$

Fix $R > 0$ and choose $\lambda = \varepsilon^2$. Integrating with respect to the measure $\mathbb{P}_\pi$, we have (the sup below is taken over $0 \le t, s \le T$)

(46)

$$\begin{aligned}&\mathbb{E}_\pi\Big[\sup_{|t-s|\le\alpha}\Big|2\varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2} \boldsymbol{b}_i(Y_r)\,dr\Big| \ge R\Big] \\ &\le 20R^{-2}(1+T)\varepsilon^2\big(|\underline{\boldsymbol{v}}^n_{\varepsilon^2}|_2^2 + |\overline{\boldsymbol{v}}^n_{\varepsilon^2}|_2^2\big) + 10R^{-2}T/\varepsilon^2|D_t\overline{\boldsymbol{v}}^n_{\varepsilon^2} - D_t\underline{\boldsymbol{v}}^n_{\varepsilon^2}|_2^2 \\ &+5\varepsilon^2\mathbb{E}_\pi\Big[\sup_{|t-s|\le\alpha}|\overline{\mathcal{M}}^{n,\varepsilon^2}_{t/\varepsilon^2} - \overline{\mathcal{M}}^{n,\varepsilon^2}_{s/\varepsilon^2}|^2 \ge R^2\Big] + 5\varepsilon^2\mathbb{E}_\pi\Big[\sup_{|t-s|\le\alpha}|\underline{\mathcal{M}}^{n,\varepsilon^2}_{s/\varepsilon^2} - \underline{\mathcal{M}}^{n,\varepsilon^2}_{t/\varepsilon^2}|^2 \ge R^2\Big].\end{aligned}$$

We are now going to explain how to choose n to make each term of the above right-hand side go to 0 as ε goes to 0.
Since $(\lambda - \boldsymbol{S})\boldsymbol{w}_\lambda = \boldsymbol{b}_i$ and $(\lambda - \boldsymbol{L})\overline{\boldsymbol{v}}^n_\lambda = \boldsymbol{b}_i - \boldsymbol{A}^n_\lambda$, we can subtract these equalities and obtain, for each $\boldsymbol{\varphi} \in \mathbb{F}$, $B^0_\lambda(\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda, \boldsymbol{\varphi}) = \boldsymbol{T}_H\big(\boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda, \boldsymbol{\varphi}\big)$ (remind of the definition of B^0_λ and $\boldsymbol{T}_H$ in (19) and (17)). Choosing $\boldsymbol{\varphi} = \boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda$, we obtain a first estimate

$$\lambda|\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda|_2^2 + (m/2)\|\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda\|_1^2 \le (2m)^{-1}(C_1^H)^2\|\boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda\|_1^2. \tag{47}$$

Following Proposition 5.4, we can differentiate the equation $B^0_\lambda(\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda, \boldsymbol{\varphi}) = \boldsymbol{T}_H\big(\boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda, \boldsymbol{\varphi}\big)$ with respect to the time variable. So we have, for each $\boldsymbol{\varphi} \in \mathbb{H}$, $B^0_\lambda(D_t\boldsymbol{w}_\lambda - D_t\overline{\boldsymbol{v}}^n_\lambda, \boldsymbol{\varphi}) = \boldsymbol{T}_H\big(D_t\boldsymbol{w}_\lambda - D_t\boldsymbol{\psi}^n_\lambda, \boldsymbol{\varphi}\big) + \partial_t\boldsymbol{T}_H\big(\boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda, \boldsymbol{\varphi}\big) - [\partial_t\boldsymbol{T}_a + \partial_t\boldsymbol{T}_H]\big(\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda, \boldsymbol{\varphi}\big)$. Choosing $\boldsymbol{\varphi} = D_t\boldsymbol{w}_\lambda - D_t\boldsymbol{\psi}^n_\lambda$, we obtain a second estimate

$$\begin{aligned}&\lambda|D_t\boldsymbol{w}_\lambda - D_t\overline{\boldsymbol{v}}^n_\lambda|_2^2 + (m/2)\|D_t\boldsymbol{w}_\lambda - D_t\overline{\boldsymbol{v}}^n_\lambda\|_1^2 \\ &\quad\le (2m)^{-1}\big(C_1^H\|D_t\boldsymbol{w}_\lambda - D_t\boldsymbol{\psi}^n_\lambda\|_1 + C_2^H\|\boldsymbol{w}_\lambda - \boldsymbol{\psi}^n_\lambda\|_1 + (C_2^a + C_2^H)\|\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}^n_\lambda\|_1\big)^2.\end{aligned} \tag{48}$$

Likewise, (47) and (48) remain valid for $\underline{\boldsymbol{v}}_\lambda^n$ instead of $\overline{\boldsymbol{v}}_\lambda^n$. For each fixed $\lambda > 0$, we can then choose $n_\lambda \in \mathbb{N}$ large enough to ensure that $|\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 + \|\boldsymbol{w}_\lambda - \overline{\boldsymbol{v}}_\lambda^{n_\lambda}\|_1^2 + \lambda^{-1}|D_t\boldsymbol{w}_\lambda - D_t\overline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 \leq \lambda$ and $|\boldsymbol{w}_\lambda - \underline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 + \|\boldsymbol{w}_\lambda - \underline{\boldsymbol{v}}_\lambda^{n_\lambda}\|_1^2 + \lambda^{-1}|D_t\boldsymbol{w}_\lambda - D_t\underline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 \leq \lambda$. From Proposition 5.8, there exists $\boldsymbol{\zeta} \in (L^2(\Omega))^d$ such that $\lambda|\boldsymbol{w}_\lambda|_2^2 + |\nabla^\sigma \boldsymbol{w}_\lambda - \boldsymbol{\zeta}|_2 \to 0$ as λ goes to 0. From (47) (with $n = n_\lambda$), $\lambda|\overline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 + \lambda|\underline{\boldsymbol{v}}_\lambda^{n_\lambda}|_2^2 \to 0$ as λ goes to 0. Hence, choosing $n = n_{\varepsilon^2}$ in (46), all the terms in the right-hand side except the martingale terms converge to 0 as ε goes to 0.
Let us now focus on the martingale terms. In order to prove the tightness of the two martingales, it is sufficient to prove the tightness of their brackets (see [6] Theorem 4.13), which respectively match $\varepsilon^2 \int_0^{t/\varepsilon^2} |\nabla^\sigma \overline{\boldsymbol{v}}_{\varepsilon^2}^{n_{\varepsilon^2}}(Y_r)|^2\, dr$ and $\varepsilon^2 \int_0^{t/\varepsilon^2} |\nabla^\sigma \underline{\boldsymbol{v}}_{\varepsilon^2}^{n_{\varepsilon^2}}(Y_r)|^2\, dr$. Note that $|\nabla^\sigma \overline{\boldsymbol{v}}_{\varepsilon^2}^{n_{\varepsilon^2}} - \boldsymbol{\zeta}|_2 \to 0$ as ε tends to 0 so that the process $t \mapsto \varepsilon^2 \int_0^{t/\varepsilon^2} |\nabla^\sigma \overline{\boldsymbol{v}}_{\varepsilon^2}^{n_{\varepsilon^2}}(Y_r)|^2\, dr$ has the same limit in $C([0,T];\mathbb{R})$ as the process $t \mapsto \varepsilon^2 \int_0^{t/\varepsilon^2} |\boldsymbol{\zeta}(Y_r)|^2\, dr$. Finally, for each fixed t, Theorem 7.1 proves that $\varepsilon^2 \int_0^{t/\varepsilon^2} |\boldsymbol{\zeta}(Y_r)|^2\, dr$ converges to the deterministic non-decreasing process $t \int_\Omega |\boldsymbol{\zeta}|_2^2\, d\pi$ in L^1 under the measure $\mathbb{P}_\pi$. Then Theorem 3.37 in [6] says that the brackets are tight in $C([0,T];\mathbb{R})$. The same arguments remain valid for the brackets of $\underline{\mathcal{M}}^{n_{\varepsilon^2},\varepsilon^2}$. Hence, the right-hand side in (46) converges to 0 as ε goes to 0 and the tightness of $t \mapsto \varepsilon X_{t/\varepsilon^2}^\omega$ follows. $\square$

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